Chapter 1.1. Numbers, Functions, Graphs

1. Use a calculator to find a rational number \( r \) such that \(| r - \pi^2 | < 10^{-4}\).

2. Which of (a)–(f) are true for \( a = -3 \) and \( b = 2\)?
   (a) \( a < b \)   (b) \( |a| < |b| \)   (c) \( ab > 0 \)
   (d) \( 3a < 3b \)   (e) \( -4a < -4b \)   (f) \( 1 \leq a \leq 5 \)

In Exercises 3–8, express the interval in terms of an inequality involving absolute value.

3. \([−2, 2]\)  
4. \((-4, 4)\)  
5. \((0, 4)\)  
6. \([-4, 0]\)  
7. \([1, 5]\)  
8. \((-2, 8)\)

In Exercises 9–12, write the inequality in the form \( a < x < b \).

9. \(|x| < 8\)  
10. \(|x − 12| < 8\)  
11. \(|2x + 1| < 5\)  
12. \(|3x − 4| < 2\)

In Exercises 13–18, express the set of numbers \( x \) satisfying the given condition as an interval.

13. \(|x| < 4\)  
14. \(|x| \leq 9\)  
15. \(|x − 4| < 2\)  
16. \(|x + 7| < 2\)  
17. \(|4x − 1| \leq 8\)  
18. \(|3x + 5| \leq 1\)

In Exercises 19–22, describe the set as a union of finite or infinite intervals.

19. \( \{x: |x − 4| > 2\} \)  
20. \( \{x: |2x + 4| > 3\} \)  
21. \( \{x: |x^2 − 1| > 2\} \)  
22. \( \{x: |x^2 + 2x| > 2\} \)

23. Match (a)–(f) with (i)–(vi).
   (a) \( a > 3 \)   (b) \( |a − 5| < \frac{1}{3} \)   (c) \( \left| a − \frac{1}{3} \right| < 5 \)
   (d) \( |a| > 5 \)   (e) \( |a − 4| < 3 \)   (f) \( 1 \leq a \leq 5 \)

(i) \( a \) lies to the right of 3.
(ii) \( a \) lies between 1 and 7.
(iii) The distance from \( a \) to 5 is less than \( \frac{1}{3} \).
(iv) The distance from \( a \) to 3 is at most 2.
(v) \( a \) is less than 5 units from \( \frac{1}{3} \).
(vi) \( a \) lies either to the left of −5 or the right of 5.

24. Describe \( \left\{ x: \frac{x}{x + 1} < 0 \right\} \) as an interval.

25. Describe \( \{x: x^2 + 2x < 3\} \) as an interval. Hint: Plot \( y = x^2 + 2x − 3 \).

26. Describe the set of real numbers satisfying \(|x − 3| = |x − 2| + 1\) as a half–infinite interval.

27. Show that if \( a > b \), then \( b^{-1} > a^{-1} \), provided that \( a \) and \( b \) have the same sign. What happens if \( a > 0 \) and \( b < 0 \)?

28. Which \( x \) satisfy both \(|x − 3| < 2\) and \(|x − 5| < 1\)?

29. Show that if \(|a − 5| < \frac{1}{2}\) and \(|b − 8| < \frac{1}{2}\), then \(|(a + b) − 13| < 1\). Hint: Use the triangle inequality.

30. Suppose that \(|x − 4| = 1\).
   (a) What is the maximum possible value of \(|x + 4|\)?
   (b) Show that \(|x^2 − 16| = 9\).

31. Suppose that \(|a − 6| \leq 2\) and \(|b| \leq 3\).
   (a) What is the largest possible value of \(|a + b|\)?
   (b) What is the smallest possible value of \(|a + b|\)?

32. Prove that \(|x| − |y| \leq |x − y|\). Hint: Apply the triangle inequality to \( y \) and \( x − y \).
33. Express $r_1 = 0.\overline{27}$ as a fraction. **Hint:** $100r_1 - r_1$ is an integer. Then express $r_2 = 0.2666\ldots$ as a fraction.

34. Represent $\frac{1}{7}$ and $\frac{4}{27}$ as repeating decimals.

35. The text states: *If the decimal expansions of numbers $a$ and $b$ agree to $k$ places, then $|a - b| \leq 10^{-k}$. Show that the converse is false: For all $k$ there are numbers $a$ and $b$ whose decimal expansions do not agree at all but $|a - b| = 10^{-k}$.***

36. Plot each pair of points and compute the distance between them:
   (a) (1, 4) and (3, 2)  
   (b) (2, 1) and (2, 4)  
   (c) (0, 0) and (−2, 3)  
   (d) (−3, −3) and (−2, 3)

37. Find the equation of the circle with center (2, 4):
   (a) with radius $r = 3$.  
   (b) that passes through (1, −1).

38. Find all points with integer coordinates located at a distance 5 from the origin. Then find all points with integer coordinates located at a distance 5 from (2, 3).

39. Determine the domain and range of the function
   $$f : \{r, s, t, u\} \rightarrow \{A, B, C, D, E\}$$
   defined by $f(r) = A, f(s) = B, f(t) = B, f(u) = E$.

40. Give an example of a function whose domain $D$ has three elements and whose range $R$ has two elements. Does a function exist whose domain $D$ has two elements and whose range $R$ has three elements?

   **In Exercises 41–48, find the domain and range of the function.**

41. $f(x) = -x$
42. $g(t) = t^4$
43. $f(x) = x^3$
44. $g(t) = \sqrt{2 - t}$
45. $f(x) = |x|$
46. $h(x) = \frac{1}{x}$
47. $f(x) = \frac{1}{x^2}$
48. $g(t) = \cos \frac{1}{t}$

   **In Exercises 49–52, determine where $f(x)$ is increasing.**

49. $f(x) = |x + 1|$
50. $f(x) = x^3$
51. $f(x) = x^4$
52. $f(x) = \frac{1}{x^4 + x^2 + 1}$

   **In Exercises 53–58, find the zeros of $f(x)$ and sketch its graph by plotting points. Use symmetry and increase/decrease information where appropriate.**

53. $f(x) = x^2 - 4$
54. $f(x) = 2x^2 - 4$
55. $f(x) = x^3 - 4x$
56. $f(x) = x^3$
57. $f(x) = 2 - x^3$
58. $f(x) = \frac{1}{(x - 1)^2 + 1}$

59. Which of these curves in is the graph of a function?

60. Determine whether the function is even, odd, or neither.
   (a) $f(x) = x^5$  
   (b) $g(t) = t^3 - t^2$
   (c) $F(t) = \frac{1}{t^4 + t^2}$
61. Determine whether the function is even, odd, or neither.

(a) \( f(t) = \frac{1}{t^4 + t + 1} - \frac{1}{t^4 - t + 1} \)

(b) \( g(t) = 2^t - 2^{-t} \)

(c) \( G(\theta) = \sin \theta + \cos \theta \)

(d) \( H(\theta) = \sin(\theta^2) \)

62. Write \( f(x) = 2x^4 - 5x^3 + 12x^2 - 3x + 4 \) as the sum of an even and an odd function.

63. Show that \( f(x) = \ln \left( \frac{1-x}{1+x} \right) \) is an odd function.

64. State whether the function is increasing, decreasing, or neither.

(a) Surface area of a sphere as a function of its radius

(b) Temperature at a point on the equator as a function of time

(c) Price of an airline ticket as a function of the price of oil

(d) Pressure of the gas in a piston as a function of volume

In Exercises 65–70, let \( f(x) \) be the function shown in Figure 27.

65. Find the domain and range of \( f(x) \)?

66. Sketch the graphs of \( f(x + 2) \) and \( f(x) + 2 \).

67. Sketch the graphs of \( f(2x), f \left( \frac{1}{2}x \right), \) and \( 2f(x) \).

68. Sketch the graphs of \( f(-x) \) and \( -f(-x) \).

69. Extend the graph of \( f(x) \) to \([-4, 4]\) so that it is an even function.

70. Extend the graph of \( f(x) \) to \([-4, 4]\) so that it is an odd function.

71. Suppose that \( f(x) \) has domain \([4, 8]\) and range \([2, 6]\). Find the domain and range of:

(a) \( f(x) + 3 \)

(b) \( f(x + 3) \)

(c) \( f(3x) \)

(d) \( 3f(x) \)

72. Let \( f(x) = x^3 \). Sketch the graph over \([-2, 2]\) of:

(a) \( f(x + 1) \)

(b) \( f(x) + 1 \)

(c) \( f(5x) \)

(d) \( 5f(x) \)

73. Suppose that the graph of \( f(x) = \sin x \) is compressed horizontally by a factor of 2 and then shifted 5 units to the right.

(a) What is the equation for the new graph?

(b) What is the equation if you first shift by 5 and then compress by 2?

(c) Verify your answers by plotting your equations.

74. Figure 28 shows the graph of \( f(x) = |x| + 1 \). Match functions (a)–(e) with their graphs (i)–(v).

(a) \( f(x - 1) \)

(b) \( -f(x) \)

(c) \( -f(x) + 2 \)

(d) \( f(x - 1) - 2 \)

(e) \( f(x + 1) \)
75. Sketch the graph of \( f(2x) \) and \( f\left(\frac{1}{2}x\right) \), where \( f(x) = |x| + 1 \) (Figure 28).

76. Find the function \( f(x) \) whose graph is obtained by shifting the parabola \( y = x^2 \) three units to the right and four units down, as in Figure 29.

77. Define \( f(x) \) to be the larger of \( x \) and \( 2 - x \). Sketch the graph of \( f(x) \). What are its domain and range? Express \( f(x) \) in terms of the absolute value function.

78. For each curve in Figure 30, state whether it is symmetric with respect to the \( y \)-axis, the origin, both, or neither.

79. Show that the sum of two even functions is even and the sum of two odd functions is odd.

80. Suppose that \( f(x) \) and \( g(x) \) are both odd. Which of the following functions are even? Odd?

(a) \( f(x)g(x) \)  
(b) \( f(x)^3 \)  
(c) \( f(x) - g(x) \)  
(d) \( \frac{f(x)}{g(x)} \)

81. Prove that the only function whose graph is symmetric with respect to both the \( y \)-axis and the origin is the function \( f(x) = 0 \).

82. Prove the triangle inequality by adding the two inequalities \( -|a| \leq a \leq |a|, -|b| \leq b \leq |b| \)

83. Show that a fraction \( r = a/b \) in lowest terms has a finite decimal expansion if and only if

\[
b = 2^n 5^m \quad \text{for some} \quad n, m \geq 0.
\]

Hint: Observe that \( r \) has a finite decimal expansion when \( 10^N r \) is an integer for some \( N \geq 0 \) (and hence \( b \) divides \( 10^N \)).

84. Let \( p = p_1 \ldots p_s \) be an integer with digits \( p_1, \ldots, p_s \). Show that \( 10^s - 1 = \frac{p}{\prod_{k=1}^{s} p_k} \). Use this to find the decimal expansion of \( r = \frac{2}{11} \). Note that
85. A function \( f(x) \) is symmetric with respect to the vertical line \( x = a \) if \( f(a - x) = f(a + x) \).

(a) Draw the graph of a function that is symmetric with respect to \( x = 2 \).

(b) Show that if \( f(x) \) is symmetric with respect to \( x = a \), then \( g(x) = f(x + a) \) is even.

86. Formulate a condition for \( f(x) \) to be symmetric with respect to the point \((a, 0)\) on the \( x\)-axis.

**Chapter 1.2, Linear/Quad functions**

In Exercises 1–4, find the slope, the \( y \)-intercept, and the \( x \)-intercept of the line with the given equation.

1. \( y = 3x + 12 \)  
2. \( y = 4 - x \)  
3. \( 4x + 9y = 3 \)  
4. \( y - 3 = \frac{3}{4}(x - 6) \)

In Exercises 5–8, find the slope of the line.

5. \( y = 3x + 2 \)  
6. \( y = 3(x - 9) + 2 \)  
7. \( 3x + 4y = 12 \)  
8. \( 3x + 4y = -8 \)

In Exercises 9–20, find the equation of the line with the given description.

9. Slope 3, \( y \)-intercept 8
10. Slope \(-2\), \( y \)-intercept 3
11. Slope 3, passes through \((7, 9)\)
12. Slope \(-5\), passes through \((0, 0)\)
13. Horizontal, passes through \((0, -2)\)
14. Passes through \((-1, 4)\) and \((2, 7)\)
15. Parallel to \( y = 3x - 4 \), passes through \((1, 1)\)
16. Passes through \((1, 4)\) and \((12, -3)\)
17. Perpendicular to \( 3x + 5y = 9 \), through \((2, 3)\)
18. Vertical, passes through \((-4, 9)\)
19. Horizontal, passes through \((8, 4)\)
20. Slope 3, \( x \)-intercept 6
21. Find the equation of the perpendicular bisector of the segment joining \((1, 2)\) and \((5, 4)\) (Figure 11). *Hint:* The midpoint \( Q \) of the segment joining \((a, b)\) and \((c, d)\) is \( \left( \frac{a + c}{2}, \frac{b + d}{2} \right) \).

22. **Intercept-Intercept Form** Show that if \( a, b \neq 0 \), then the line with \( x \)-intercept \( x = a \) and \( y \)-intercept \( y = b \) has equation (Figure 12)

\[ \frac{x}{a} + \frac{y}{b} = 1 \]

23. Find an equation of the line with \( x \)-intercept \( x = 4 \) and \( y \)-intercept \( y = 3 \).
24. Find y such that (3, y) lies on the line of slope \( m = 2 \) through (1, 4).

25. Determine whether there exists a constant \( c \) such that the line \( x + cy = 1 \):

(a) Has slope 4  
(b) Passes through (3, 1)  
(c) Is horizontal  
(d) Is vertical

26. Assume that the number \( N \) of concert tickets that can be sold at a price of \( P \) dollars per ticket is a linear function \( N(P) \) for \( 10 \leq P \leq 40 \). Determine \( N(P) \) (called the demand function) if \( N(10) = 500 \) and \( N(40) = 0 \). What is the decrease \( \Delta N \) in the number of tickets sold if the price is increased by \( \Delta P = 5 \) dollars?

27. Materials expand when heated. Consider a metal rod of length \( L_0 \) at temperature \( T_0 \). If the temperature is changed by an amount \( \Delta T \), then the rod’s length changes by \( \Delta L = \alpha L_0 \Delta T \), where \( \alpha \) is the thermal expansion coefficient. For steel, \( \alpha = 1.24 \times 10^{-5} \) °C\(^{-1} \).

(a) A steel rod has length \( L_0 = 40 \) cm at \( T_0 = 40 \) °C. Find its length at \( T = 90 \) °C.

(b) Find its length at \( T = 50 \) °C if its length at \( T_0 = 100 \) °C is 65 cm.

(c) Express length \( L \) as a function of \( T \) if \( L_0 = 65 \) cm at \( T_0 = 100 \) °C.

28. Do \((0.5, 1), (1, 1.2), (2, 2)\) lie on a line?

29. Find \( b \) so \((2, -1), (3, 2), \) and \((b, 5)\) lie on a line.

30. Find an expression for the velocity \( v \) as a linear function of \( t \) that matches the following data.

<table>
<thead>
<tr>
<th>( (T_1, P_1) )</th>
<th>( (T_2, P_2) )</th>
<th>( \frac{\Delta P}{\Delta T} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(40, 1365.80)</td>
<td>(45, 1385.40)</td>
<td>( \frac{1385.40 - 1365.80}{45 - 40} = 3.92 )</td>
</tr>
<tr>
<td>(45, 1385.40)</td>
<td>(55, 1424.60)</td>
<td>( \frac{1424.60 - 1385.40}{55 - 45} = 3.92 )</td>
</tr>
<tr>
<td>(55, 1424.60)</td>
<td>(70, 1483.40)</td>
<td>( \frac{1483.40 - 1424.60}{70 - 55} = 3.92 )</td>
</tr>
<tr>
<td>(70, 1483.40)</td>
<td>(80, 1522.60)</td>
<td>( \frac{1522.60 - 1483.40}{80 - 70} = 3.92 )</td>
</tr>
</tbody>
</table>

31. The period \( T \) of a pendulum is measured for pendulums of several different lengths \( L \). Based on the following data, does \( T \) appear to be a linear function of \( L \)?

<table>
<thead>
<tr>
<th>( t ) (s)</th>
<th>( v ) (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>39.2</td>
</tr>
<tr>
<td>2</td>
<td>58.6</td>
</tr>
<tr>
<td>4</td>
<td>78</td>
</tr>
<tr>
<td>6</td>
<td>97.4</td>
</tr>
</tbody>
</table>

32. Show that \( f(x) \) is linear of slope \( m \) if and only if

\[ f(x + h) - f(x) = mh \] (for all \( x \) and \( h \))

33. Find the roots of the quadratic polynomials:

(a) \( 4x^2 - 3x - 1 \)  
(b) \( x^2 - 2x - 1 \)

In Exercises 34–41, complete the square and find the minimum or maximum value of the quadratic function.

34. \( y = x^2 + 2x + 5 \)  
35. \( y = x^2 - 6x + 9 \)

36. \( y = -9x^2 + x \)  
37. \( y = x^2 + 6x + 2 \)

38. \( y = 2x^2 - 4x - 7 \)  
39. \( y = -4x^2 + 3x + 8 \)

40. \( y = 3x^2 + 12x - 5 \)  
41. \( y = 4x - 12x^2 \)

42. Sketch the graph of \( y = x^2 - 6x + 8 \) by plotting the roots and the minimum point.
43. Sketch \( y = x^2 + 4x + 6 \) by plotting the minimum point, the y-intercept, and one other point.

44. If the alleles \( A \) and \( B \) of the cystic fibrosis gene occur in a population with frequencies \( p \) and \( 1 - p \) (where \( p \) is a fraction between 0 and 1), then the frequency of heterozygous carriers (carriers with both alleles) is \( 2p(1 - p) \). Which value of \( p \) gives the largest frequency of heterozygous carriers?

45. For which values of \( c \) does \( f(x) = x^2 + cx + 1 \) have a double root? No real roots?

46. Let \( f(x) \) be a quadratic function and \( c \) a constant. Which of the following statements is correct? Explain graphically.

(a) There is a unique value of \( c \) such that \( y = f(x) - c \) has a double root.

(b) There is a unique value of \( c \) such that \( y = f(x - c) \) has a double root.

47. Prove that \( \frac{x}{x^2} + \frac{1}{x} \geq 2 \) for all \( x > 0 \). Hint: Consider \((x^{1/2} - x^{-1/2})^2\).

48. Let \( a, b > 0 \). Show that the geometric mean \( \sqrt{ab} \) is not larger than the arithmetic mean \( \frac{a + b}{2} \). Hint: Use a variation of the hint given in Exercise 47.

49. If objects of weights \( x \) and \( w_1 \) are suspended from the balance in Figure 13(A), the cross-beam is horizontal if \( bx = aw_1 \). If the lengths \( a \) and \( b \) are known, we may use this equation to determine an unknown weight \( x \) by selecting \( w_1 \) such that the cross-beam is horizontal. If \( a \) and \( b \) are not known precisely, we might proceed as follows. First balance \( x \) by \( w_1 \) on the left as in (A). Then switch places and balance \( x \) by \( w_2 \) on the right as in (B). The average \( \bar{x} = \frac{1}{2}(w_1 + w_2) \) gives an estimate for \( x \). Show that \( \bar{x} \) is greater than or equal to the true weight \( x \).

50. Find numbers \( x \) and \( y \) with sum 10 and product 24. Hint: Find a quadratic polynomial satisfied by \( x \).

51. Find a pair of numbers whose sum and product are both equal to 8.

52. Show that the parabola \( y = x^2 \) consists of all points \( P \) such that \( d_1 = d_2 \), where \( d_1 \) is the distance from \( P \) to \( \left(0, \frac{1}{4}\right)\), and \( d_2 \) is the distance from \( P \) to the line \( y = -\frac{1}{4} \) (Figure 14).

53. Show that if \( f(x) \) and \( g(x) \) are linear, then so is \( f(x) + g(x) \). Is the same true of \( f(x)g(x) \)?

54. Show that if \( f(x) \) and \( g(x) \) are linear functions such that \( f(0) = g(0) \) and \( f(1) = g(1) \), then \( f(x) = g(x) \).

55. Show that \( \frac{\Delta y}{\Delta x} \) for the function \( f(x) = x^2 \) over the interval \([x_1, x_2]\) is not a constant, but depends on the interval. Determine the exact dependence of \( \Delta y/\Delta x \) on \( x_1 \) and \( x_2 \).
56. Use Eq. (2) to derive the quadratic formula for the roots of \(ax^2 + bx + c = 0\).

57. Let \(a, c \neq 0\) Show that the roots of 
\[ax^2 + bx + c = 0\]
and 
\[cx + bx + a = 0\]
are reciprocals of each other.

58. Show, by completing the square, that the parabola 
\[y = ax^2 + bx + c\]
is congruent to 
\[y = ax^2\]
by a vertical and horizontal translation.

59. Prove Viéte’s Formulas: The quadratic polynomial with \(\alpha\) and \(\beta\) as roots is 
\[x^2 + bx + c\],
where \(b = -\alpha - \beta\) and \(c = \alpha \beta\).

Chapter 1.3, Basic Function Classes

In Exercises 1–12, determine the domain of the function.

1. \(f(x) = x^{1/4}\)  
2. \(g(t) = t^{2/3}\)

3. \(f(x) = x^3 + 3x - 4\)  
4. \(h(z) = z^3 + z^{-3}\)

5. \(g(t) = \frac{1}{t + 2}\)  
6. \(f(x) = \frac{1}{x^2 + 4}\)

7. \(G(u) = \frac{1}{u^2 - 4}\)  
8. \(f(x) = \frac{\sqrt{x}}{x^2 - 9}\)

9. \(f(x) = x^{-4} + (x - 1)^{-3}\)  
10. \(F(s) = \sin\left(\frac{s}{s + 1}\right)\)

11. \(g(y) = 10\sqrt{y} + y^{-1}\)  
12. \(f(x) = \frac{x + x^{-1}}{(x - 3)(x + 4)}\)

In Exercises 13–24, identify each of the following functions as polynomial, rational, algebraic, or transcendental.

13. \(f(x) = 4x^3 + 9x^2 - 8\)  
14. \(f(x) = x^{-4}\)

15. \(f(x) = \sqrt{x}\)  
16. \(f(x) = \sqrt{1 - x^2}\)

17. \(f(x) = \frac{x^2}{x + \sin x}\)  
18. \(f(x) = 2^x\)

19. \(f(x) = \frac{2x^3 + 3x}{9 - 7x^2}\)  
20. \(f(x) = \frac{3x - 9x^{-1/2}}{9 - 7x^2}\)

21. \(f(x) = \sin(x^2)\)  
22. \(f(x) = \frac{x}{\sqrt{x} + 1}\)

23. \(f(x) = x^2 + 3x^{-1}\)  
24. \(f(x) = \sin(3^x)\)

25. Is \(f(x) = 2^x\) a transcendental function?

26. Show that \(f(x) = x^2 + 3x^{-1}\) and \(g(x) = 3x^3 - 9x + x^{-2}\) are rational functions—that is, quotients of polynomials.

In Exercises 27–34, calculate the composite functions \(f \circ g\) and \(g \circ f\), and determine their domains.

27. \(f(x) = \sqrt{x}, \ g(x) = x + 1\)

28. \(f(x) = \frac{1}{x}, \ g(x) = x^{-4}\)

29. \(f(x) = 2^x, g(x) = x^2\)

30. \(f(x) = |x|, g(\theta) = \sin \theta\)

31. \(f(\theta) = \cos \theta, g(x) = x^3 + x^2\)

32. \(f(x) = \frac{1}{x^2 + 1}, \ g(x) = x^{-2}\)

33. \(f(t) = \frac{1}{\sqrt{t}}, \ g(t) = -t^2\)

34. \(f(t) = \sqrt{t}, \ g(t) = 1 - t^3\)

35. The population (in millions) of a country as a function of time \(t\) (years) is 
\[P(t) = 30.20^{0.12t}\]. Show that the population doubles every 10 years. Show more generally that for any positive constants \(a\) and \(k\), the function \(g(t) = a2^{kt}\) doubles after \(\frac{1}{k}\) years. \(x + 1\)

36. Find all values of \(c\) such that 
\[f(x) = \frac{x + 1}{x^2 + 2cx + 4}\] has domain \(\mathbb{R}\)
In Exercises 37–43, we define the first difference \( \delta f \) of a function \( f(x) \) by \( \delta f(x) = f(x + 1) - f(x) \).

37. Show that if \( f(x) = x^2 \), then \( \delta f(x) = 2x + 1 \). Calculate \( \delta f \) for \( f(x) = x \) and \( f(x) = x^3 \).

38. Show that \( \delta(10^x) = 9 \cdot 10^x \) and, more generally, that \( \delta(b^x) = (b - 1)b^x \).

39. Show that for any two functions \( f \) and \( g \), \( \delta(f + g) = \delta f + \delta g \) and \( \delta(cf) = c\delta(f) \), where \( c \) is any constant.

40. Suppose we can find a function \( P(x) \) such that \( \delta P = (x + 1)^k \) and \( P(0) = 0 \). Prove that \( P(1) = 1^k \), \( P(2) = 1^k + 2^k \), and, more generally, for every whole number \( n \),

\[
P(n) = 1^k + 2^k + \cdots + n^k
\]

41. First show that \( P(x) = \frac{x(x + 1)}{2} \) satisfies \( \delta P = (x + 1) \). Then apply Exercise 40 to conclude that

\[
1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}
\]

42. Calculate \( \delta(x^3) \), \( \delta(x^2) \), and \( \delta(x) \). Then find a polynomial \( P(x) \) of degree 3 such that \( \delta P = (x + 1)^2 \) and \( P(0) = 0 \). Conclude that \( P(n) = 1^2 + 2^2 + \cdots + n^2 \).

Chapter 1.4, Trigonometric Functions

1. Find the angle between 0 and \( 2\pi \) equal to \( 13\pi/4 \).

2. Describe \( \theta = \pi/6 \) by an angle of negative radian measure.

3. Convert from radians to degrees:
   (a) \( \frac{\pi}{3} \)  (b) \( \frac{5}{12} \)  (c) \( \frac{3\pi}{4} \)

4. Convert from degrees to radians:
   (a) \( 1^\circ \)  (b) \( 30^\circ \)  (c) \( 25^\circ \)  (d) \( 120^\circ \)

5. Find the lengths of the arcs subtended by the angles \( \theta \) and \( \phi \) radians in Figure 20.

6. Calculate the values of the six standard trigonometric functions for the angle \( \theta \) in Figure 21.

7. Fill in the remaining values of \( (\cos \theta, \sin \theta) \) for the points in Figure 22.

8. Find the values of the six standard trigonometric functions at \( \theta = 11\pi/6 \).

In Exercises 9–14, use Figure 22 to find all angles between 0 and \( 2\pi \) satisfying the given condition.

9. \( \cos \theta = \frac{1}{2} \)  10. \( \tan \theta = 1 \)

11. \( \tan \theta = -1 \)  12. \( \csc \theta = 2 \)

13. \( \sin x = \frac{\sqrt{3}}{2} \)  14. \( \sec t = 2 \)
15. Fill in the following table of values:

- Periodicity: \( \sin(\theta + 2\pi) = \sin \theta, \quad \cos(\theta + 2\pi) = \cos \theta \)
- Parity: \( \sin(-\theta) = -\sin \theta, \quad \cos(-\theta) = \cos \theta \)
- Basic identity: \( \sin^2 \theta + \cos^2 \theta = 1 \)

16. Complete the following table of signs:

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \pi/6 )</th>
<th>( \pi/4 )</th>
<th>( \pi/3 )</th>
<th>( \pi/2 )</th>
<th>( 2\pi/3 )</th>
<th>( 3\pi/4 )</th>
<th>( 5\pi/6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tan \theta )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \sec \theta )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

17. Show that if \( \tan \theta = c \) and \( 0 \leq \theta < \pi/2 \), then \( \cos \theta = 1/\sqrt{1 + c^2} \). *Hint:* Draw a right triangle whose opposite and adjacent sides have lengths \( c \) and 1.

18. Suppose that \( \cos \theta = \frac{1}{3} \).

(a) Show that if \( 0 \leq \theta < \pi/2 \), then \( \sin \theta = 2\sqrt{2}/3 \) and \( \tan \theta = 2\sqrt{2} \).

(b) Find \( \sin \theta \) and \( \tan \theta \) and \( \tan \theta \) if \( 3\pi/2 \leq \theta < 2\pi \).

In Exercises 19–24, assume that \( 0 \leq \theta < \pi/2 \).

19. Find \( \sin \theta \) and \( \tan \theta \) if \( \cos \theta = \frac{5}{13} \).

20. Find \( \cos \theta \) and \( \tan \theta \) if \( \sin \theta = \frac{3}{5} \).

21. Find \( \sin \theta \), \( \sec \theta \), and \( \cot \theta \) if \( \tan \theta = \frac{2}{7} \).

22. Find \( \sin \theta \), \( \cos \theta \), and \( \sec \theta \) if \( \cot \theta = 4 \).

23. Find \( \cos 2\theta \) if \( \sin \theta = \frac{1}{3} \).

24. Find \( \sin 2\theta \) and \( \cos 2\theta \) if \( \tan \theta = \sqrt{2} \).

25. Find \( \cos \theta \) and \( \tan \theta \) if \( \sin \theta = 0.4 \) and \( \pi/2 \leq \theta < \pi \).

26. Find \( \cos \theta \) and \( \sin \theta \) if \( \tan \theta = 4 \) and \( \pi \leq \theta = < 3\pi/2 \).

27. Find \( \cos \theta \) if \( \cot \theta = \frac{4}{3} \) and \( \sin \theta < 0 \).

28. Find \( \tan \theta \) if \( \sec \theta = \sqrt{3} \) and \( \sin \theta < 0 \).

29. Find the values of \( \sin \theta \), \( \cos \theta \), and \( \tan \theta \) for the angles corresponding to the eight points in Figure 23(A) and (B).

30. Refer to Figure 24(A). Express the functions \( \sin \theta \), \( \tan \theta \), and \( \csc \theta \) in terms of \( c \).

31. Refer to Figure 24(B). Compute \( \cos \psi \), \( \sin \psi \), \( \cot \psi \), and \( \csc \psi \).

32. Express \( \cos \left( \theta + \frac{\pi}{2} \right) \) and \( \sin \left( \theta + \frac{\pi}{2} \right) \) in terms of \( \cos \theta \) and \( \sin \theta \).

*Hint:* Find the relation between the coordinates \((a, b)\) and \((c, d)\) in Figure 25.
33. Use the addition formula to compute \( \cos \left( \frac{\pi}{3} + \frac{\pi}{4} \right) \) exactly.

34. Use the addition formula to compute \( \sin \left( \frac{\pi}{3} - \frac{\pi}{4} \right) \) exactly.

In Exercises 35–38, sketch the graph over \([0, 2\pi]\).

35. \( 2 \sin 4\theta \)  
36. \( \cos \left( 2 \left( \theta - \frac{\pi}{2} \right) \right) \)  
37. \( \cos \left( 2\theta - \frac{\pi}{2} \right) \)  
38. \( \sin \left( 2 \left( \theta - \frac{\pi}{2} \right) + \pi \right) + 2 \)

39. How many points lie on the intersection of the horizontal line \( y = c \) and the graph of \( y = \sin x \) for \( 0 \leq x < 2\pi \)? Hint: The answer depends on \( c \).

40. How many points lie on the intersection of the horizontal line \( y = c \) and the graph of \( y = \tan x \) for \( 0 \leq x < 2\pi \)?

In Exercises 41–44, solve for \( 0 \leq \theta < 2\pi \) (see Example 4).

41. \( \sin 2\theta + \sin 3\theta = 0 \)  
42. \( \sin \theta = \sin 2\theta \)  
43. \( \cos 4\theta + \cos 2\theta = 0 \)  
44. \( \sin \theta = \cos 2\theta \)

In Exercises 45–54, derive the identity using the identities listed in this section.

45. \( \cos 2\theta = 2\cos^2 \theta - 1 \)  
46. \( \cos^2 \left( \frac{\theta}{2} \right) = \frac{1 + \cos \theta}{2} \)

47. \( \sin \left( \frac{\theta}{2} \right) = \sqrt{\frac{1 - \cos \theta}{2}} \)  
48. \( \sin(\theta + \pi) = -\sin \theta \)  
49. \( \cos(\theta + \pi) = -\cos \theta \)  
50. \( \tan x = \cot \left( \frac{\pi}{2} - x \right) \)

51. \( \tan(\pi - \theta) = -\tan \theta \)  
52. \( \tan 2x = \frac{2\tan x}{1 - \tan^2 x} \)

53. \( \tan x = \frac{-\sin 2x}{1 + \cos 2x} \)  
54. \( \sin^2 x \cos^2 x = \frac{1 - \cos 4x}{8} \)

55. Use Exercise 48 and 49 to show that \( \tan \theta \) and \( \cot \theta \) are periodic with period \( \pi \).

56. Use trigonometric identities to compute \( \cos \frac{\pi}{15} \), noting that \( \frac{\pi}{15} = \frac{\pi}{2} \left( \frac{2}{3} - \frac{\pi}{5} \right) \).

57. Use the Law of Cosines to find the distance from \( P \) to \( Q \) in Figure 26.

58. Use Figure 27 to derive the Law of Cosines from the Pythagorean Theorem.

59. Use the addition formula to prove \( \cos 3\theta = 4\cos^3 \theta - 3\cos \theta \).

60. Use the addition formulas for sine and cosine to prove

\[
\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b} \]

\[
\cot(a - b) = \frac{\cot a \cot b + 1}{\cot b - \cot a} \]

61. Let \( \theta \) be the angle between the line \( y = mx + b \) and the \( x \)-axis [Figure 28(A)]. Prove that \( m = \tan \theta \).
62. Let \( L_1 \) and \( L_2 \) be the lines of slope \( m_1 \) and \( m_2 \) [Figure 28(B)]. Show that the angle \( \theta \) between \( L_1 \) and \( L_2 \) satisfies
\[
\cot \theta = \frac{m_2m_1 + 1}{m_2 - m_1}.
\]

63. **Perpendicular Lines** Use Exercise 62 to prove that two lines with nonzero slopes \( m_1 \) and \( m_2 \) are perpendicular if and only if \( m_2 = -1/m_1 \).

64. Apply the double-angle formula to prove:

(a) \[
\cos \frac{\pi}{8} = \frac{1}{2} \sqrt{2 + \sqrt{2}}
\]

(b) \[
\cos \frac{\pi}{16} = \frac{1}{2} \sqrt{2 + \sqrt{2 + \sqrt{2}}}
\]

Guess the values of \( \cos \frac{\pi}{32} \) and of \( \cos \frac{\pi}{2^n} \) for all \( n \).

**Chapter 1.5**, Inverse Functions

1. Show that \( f(x) = 7x - 4 \) is invertible and find its inverse.

2. Is \( f(x) = x^2 + 2 \) one-to-one? If not, describe a domain on which it is one-to-one.

3. What is the largest interval containing zero on which \( f(x) = \sin x \) is one-to-one?

4. Show that \( f(x) = \frac{x - 2}{x + 3} \) is invertible and find its inverse.

(a) What is the domain of \( f(x) \) and range of \( f^{-1}(x) \)?

(b) What is the domain of \( f^{-1}(x) \) and range of \( f(x) \)?

5. Verify that \( f(x) = x^3 + 3 \) and \( g(x) = (x - 3)^{1/3} \) are inverses by showing that \( f(g(x)) = x \) and \( g(f(x)) = x \).

6. Repeat Exercise 5 for \( f(t) = \frac{t + 1}{t - 1} \) and \( g(t) = \frac{t + 1}{t - 1} \).

7. The escape velocity from a planet of radius \( R \) is
\[
v(R) = \sqrt{\frac{2GM}{R}}
\]
where \( G \) is the universal gravitational constant and \( M \) is the mass. Find the inverse of \( v(R) \) expressing \( R \) in terms of \( v \).

In Exercises 8–15, find a domain on which \( f \) is one-to-one and a formula for the inverse of \( f \) restricted to this domain. Sketch the graphs of \( f \) and \( f^{-1} \).

8. \( f(x) = 3x - 2 \)

9. \( f(x) = 4 - x \)

10. \( f(x) = \frac{1}{x + 1} \)

11. \( f(x) = \frac{1}{7x - 3} \)

12. \( f(s) = \frac{1}{s^2} \)

13. \( f(x) = \frac{1}{\sqrt{x^2 + 1}} \)

14. \( f(z) = z^3 \)

15. \( f(x) = \sqrt{x^3 + 9} \)

16. For each function shown in, sketch the graph of the inverse (restrict the function’s domain if necessary).

17. Which of the graphs below is the graph of a function satisfying \( f^{-1} = f \)?
18. Let \( n \) be a nonzero integer. Find a domain on which \( f(x) = (1 - x^n)^{1/n} \) coincides with its inverse. Hint: The answer depends on whether \( n \) is even or odd.

19. Let \( f(x) = x^7 + x + 1 \).

(a) Show that \( f^{-1} \) exists (but do not attempt to find it). Hint: Show that \( f \) is increasing.

(b) What is the domain of \( f^{-1} \)?

(c) Find \( f^{-1}(3) \).

20. Show that \( f(x) = (x^2 + 1)^{-1} \) is one-to-one on \((-\infty, 0]\), and find a formula for \( f^{-1} \) on this domain.

21. Let \( f(x) = x^2 - 2x \). Determine a domain on which \( f^{-1} \) exists, and find a formula for \( f^{-1} \) for this domain of \( f \).

22. Show that \( f(x) = x + x^{-1} \) is one-to-one on \([1, \infty)\), and find the corresponding inverse \( f^{-1} \). What is the domain of \( f^{-1} \)?

In Exercises 23–28, evaluate without using a calculator.

23. \( \cos^{-1} 1 \)

24. \( \sin^{-1} \frac{1}{2} \)

25. \( \cot^{-1} 1 \)

26. \( \sec^{-1} \frac{2}{\sqrt{3}} \)

27. \( \tan^{-1} \sqrt{3} \)

28. \( \sin^{-1} (-1) \)

In Exercises 29–38, compute without using a calculator.

29. \( \sin^{-1} \left( \sin \frac{\pi}{3} \right) \)

30. \( \sin^{-1} \left( \sin \frac{4\pi}{3} \right) \)

31. \( \cos^{-1} \left( \cos \frac{3\pi}{2} \right) \)

32. \( \sin^{-1} \left( \sin \left( -\frac{5\pi}{6} \right) \right) \)

33. \( \tan^{-1} \left( \tan \frac{3\pi}{4} \right) \)

34. \( \tan^{-1} (\tan \pi) \)

35. \( \sec^{-1} (\sec 3\pi) \)

36. \( \sec^{-1} \left( \sec \frac{3\pi}{2} \right) \)

37. \( \csc^{-1} (\csc(-\pi)) \)

38. \( \cot^{-1} \left( \cot \left( -\frac{\pi}{4} \right) \right) \)

In Exercises 39–42, simplify by referring to the appropriate triangle or trigonometric identity.

39. \( \tan(\cos^{-1} x) \)

40. \( \cos(\tan^{-1} x) \)

41. \( \cot(\sec^{-1} x) \)

42. \( \cot(\sin^{-1} x) \)

In Exercises 43–50, refer to the appropriate triangle or trigonometric identity to compute the given value.

43. \( \cos(\sin^{-1} \frac{2}{3}) \)

44. \( \tan(\cos^{-1} \frac{2}{3}) \)

45. \( \tan(\sin^{-1} 0.8) \)

46. \( \cos(\cot^{-1} 1) \)

47. \( \cot(\csc^{-1} 2) \)

48. \( \tan(\sec^{-1} (-2)) \)

49. \( \cot(\tan^{-1} 20) \)

50. \( \sin(\csc^{-1} 20) \)

51. Show that if \( f(x) \) is odd and \( f^{-1}(x) \) exists, then \( f^{-1}(x) \) is odd. Show, on the other hand, that an even function does not have an inverse.

52. A cylindrical tank of radius \( R \) and length \( L \) lying horizontally as in Figure 21 is filled with oil to height \( h \). Show that the volume \( V(h) \) of oil in the tank as a function of height \( h \) is

\[
V(h) = L \left( R^2 \cos^{-1} \left( 1 - \frac{h}{R} \right) - (R - h)\sqrt{2hR - h^2} \right)
\]
Chapter 1.6, Exponential/Logarithms

1. Rewrite as a whole number (without using a calculator):

(a) \(7^5\)  
(b) \(10^2(2^7 + 5^2)\)  
(c) \((4^5)^3\)

(d) \(27^{4/3}\)  
(e) \(8^{-1/3} \cdot 8^{5/3}\)  
(f) \(3 \cdot 4^{1/4} - 12 \cdot 2^{-3/2}\)

In Exercises 2–10, solve for the unknown variable.

2. \(9^{2x} = 9^8\)

3. \(e^{2x} = e^{x+1}\)

4. \(e^{t^2} = e^{4t-3}\)

5. \(3^x = \left(\frac{1}{3}\right)^{x+1}\)

6. \((\sqrt{5})^x = 125\)

7. \(4^x = 2^{x+1}\)

8. \(b^4 = 10^{12}\)

9. \(k^{3/2} = 27\)

10. \((b^2)^{x+1} = b^{-6}\)

In Exercises 11–26, calculate without using a calculator.

11. \(\log_3 27\)

12. \(\log_5 \frac{1}{25}\)

13. \(\ln 1\)

14. \(\log_4 (5^4)\)

15. \(\log_5 (2^{5/3})\)

16. \(\log_3 (8^{5/3})\)

17. \(\log_{64} 4\)

18. \(\log_5 (49^2)\)

19. \(\log_8 2 + \log_4 2\)

20. \(\log_{25} 30 + \log_{25} \frac{5}{6}\)

21. \(\log_{14} 48 - \log_{14} 12\)

22. \(\ln(\sqrt{e} \cdot e^{7/5})\)

23. \(\ln(e^3) + \ln(e^4)\)

24. \(\log_2 \frac{4}{3} + \log_2 24\)

25. \(\log_{\frac{1}{7}} (29)\)

26. \(8^{3 \log_8 (2)}\)

27. Write as the natural log of a single expression:

(a) \(2 \ln 5 + 3 \ln 4\)  
(b) \(5 \ln(x^{1/2}) + \ln(9x)\)

28. Solve for \(x\): \(\ln(x^2 + 1) - 3 \ln x = \ln(2)\).

In Exercises 29–34, solve for the unknown.

29. \(7e^{5t} = 100\)

30. \(6e^{-4t} = 2\)

31. \(2x^2 - 2x = 8\)

32. \(e^{2x+1} = 9e^{1-t}\)

33. \(\ln(x^4) - \ln(x^2) = 2\)

34. \(\log_3 y + 3 \log_3 (y^2) = 14\)

35. Use a calculator to compute \(\sinh x\) and \(\cosh x\) for \(x = -3, 0, 5\).

36. Compute \(\sinh(\ln 5)\) and \(\tanh(3 \ln 5)\) without using a calculator.

37. Show, by producing a counterexample, that \(\ln(ab)\) is not equal to \((\ln a)(\ln b)\).

38. For which values of \(x\) are \(y = \sinh x\) and \(y = \cosh x\) increasing and decreasing?

39. Show that \(y = \tanh x\) is an odd function.

40. The population of a city (in millions) at time \(t\) (years) is \(P(t) = 2.4e^{0.06t}\), where \(t = 0\) is the year 2000. When will the population double from its size at \(t = 0\)?

41. The Gutenberg–Richter Law states that the number \(N\) of earthquakes/year worldwide of Richter magnitude at least \(M\) satisfies an approximate relation \(\log_{10} N = a - M\) for some constant \(a\). Find \(a\), assuming that there is one earthquake of magnitude \(M \geq 8\) per year. How many of magnitude \(M \geq 5\) occur per year?
42. The energy $E$ (in joules) radiated as seismic waves from an earthquake of Richter magnitude $M$ is given by the formula $\log_{10} E = 4.8 + 1.5M$.

(a) Express $E$ as a function of $M$.

(b) Show that when $M$ increases by 1, the energy increases by a factor of approximately 31.6.

43. Refer to the graphs to explain why the equation $\sinh x = t$ has a unique solution for every $t$ and why $\cosh x = t$ has two solutions for every $t > 1$.

44. Compute $\cosh x$ and $\tanh x$, assuming that $\sinh x = 0.8$.

45. Prove the addition formula for $\cosh x$.

46. Use the addition formulas to prove

$$\tanh(u + v) = \frac{\tanh u + \tanh v}{1 + \tanh u \tanh v}$$

(b) Use (a) to show that Einstein’s Law of Velocity Addition [Eq. (3)] is equivalent to

$$w = \frac{u + v}{1 + \frac{uv}{c^2}}$$

51. Prove that every function $f(x)$ can be written as a sum $f(x) = f_e(x) + f_o(x)$ of an even function $f_e(x)$ and an odd function $f_o(x)$. Express $f(x) = 5e^x + 8e^{-x}$ in terms of $\cosh x$ and $\sinh x$.

Chapter 1 REVIEW

1. Express $(4, 10)$ as a set $\{x : |x - a| < c\}$ for suitable $a$ and $c$.

2. Express as an interval:

(a) $\{x : |x - 5| < 4\}$

(b) $\{x : |5x + 3| \leq 2\}$

3. Express $\{x : 2 \leq |x - 1| \leq 6\}$ as a union of two intervals.

4. Give an example of numbers $x, y$ such that $|x| + |y| = x - y$.

5. Describe the pairs of numbers $x, y$ such that $|x + y| = x - y$.

6. Sketch the graph of $y = f(x + 2) - 1$, where $f(x) = x^2$ for $-2 \leq x \leq 2$.

In Exercises 7–10, let $f(x)$ be the function shown in Figure 1.

7. Sketch the graphs of $y = f(x) + 2$ and $y = f(x + 2)$.

8. Sketch the graphs of $y = \frac{1}{2} f(x)$ and $y = f\left(\frac{1}{2} x\right)$

9. Continue the graph of $f(x)$ to the interval $[-4, 4]$ as an even function.
10. Continue the graph of \( f(x) \) to the interval \([-4, 4]\) as an odd function.

11. \( f(x) = \sqrt{x + 1} \quad 12. \ f(x) = \frac{4}{x^4 + 1} \)
13. \( f(x) = \frac{2}{3-x} \quad 14. \ f(x) = \sqrt{x^2 - x + 5} \)

15. Determine whether the function is increasing, decreasing, or neither:
   (a) \( f(x) = 3^{-x} \)  
   (b) \( f(x) = \frac{1}{x^4 + 1} \)  
   (c) \( g(t) = t^2 + t \)  
   (d) \( g(t) = t^3 + t \)  

16. Determine whether the function is even, odd, or neither:
   (a) \( f(x) = x^4 - 3x^2 \)  
   (b) \( g(x) = \sin(x + 1) \)  
   (c) \( f(x) = 2^{-x^2} \)

17. Line passing through \((-1, 4)\) and \((2, 6)\)
18. Line passing through \((-1, 4)\) and \((-1, 6)\)
19. Line of slope 6 through \((9, 1)\)
20. Line of slope \(-\frac{3}{2}\) through \((4, -12)\)
21. Line through \((2, 3)\) parallel to \(y = 4 - x\)
22. Horizontal line through \((-3, 5)\)

23. Does the following table of market data suggest a linear relationship between price and number of homes sold during a one-year period? Explain.

   \[
   \begin{array}{|c|c|c|c|c|}
   \hline
   \text{Price (thousands of $)} & 180 & 195 & 220 & 240 \\
   \text{No. of homes sold} & 127 & 118 & 103 & 91 \\
   \hline
   \end{array}
   \]

24. Does the following table of revenue data for a computer manufacturer suggest a linear relation between revenue and time? Explain.
   \[
   \begin{align*}
   \sinh^{-1} x, & \quad \text{for all } x \\
   \coth^{-1} x, & \quad \text{for } |x| > 1 \\
   \cosh^{-1} x, & \quad \text{for } x \geq 1 \\
   \sech^{-1} x, & \quad \text{for } 0 < x \leq 1 \\
   \tanh^{-1} x, & \quad \text{for } |x| < 1 \\
   \csch^{-1} x, & \quad \text{for } x \neq 0
   \end{align*}
   \]

25. Find the roots of \( f(x) = x^4 - 4x^2 \) and sketch its graph. On which intervals is \( f(x) \) decreasing?

26. Let \( h(z) = 2z^2 + 12z + 3 \). Complete the square and find the minimum value of \( h(z) \).

27. Let \( f(x) \) be the square of the distance from the point \((2, 1)\) to a point \((x, 3x + 2)\) on the line \( y = 3x + 2 \). Show that \( f(x) \) is a quadratic function, and find its minimum value by completing the square.

28. Prove that \( x^3 + 3x + 3 \geq 0 \) for all \( x \).

29. \( y = t^4 \)
30. \( y = t^5 \)
31. \( y = \sin \frac{\theta}{2} \)
32. \( y = 10^{-x} \)
33. \( y = x^{1/3} \)
34. \( y = \frac{1}{x^2} \)
35. Show that the graph of \( y = f \left( \frac{1}{3} x - b \right) \) is obtained by shifting the graph of \( y = f \left( \frac{1}{3} x \right) \) to the right \( 3b \) units. Use this observation to sketch the graph of \( y = \left| \frac{1}{3} x - 4 \right| \).
36. Let \( h(x) = \cos x \) and \( g(x) = x^{-1} \). Compute the composite functions \( h(g(x)) \) and \( h(h(x)) \), and find their domains.

37. Find functions \( f \) and \( g \) such that the function 
\[
 f(g(x)) = (12t + 9)^4
\]

38. Sketch the points on the unit circle corresponding to the following three angles, and find the values of the six standard trigonometric functions at each angle:
\[
\begin{align*}
(a) & \quad \frac{2\pi}{3} & (b) & \quad \frac{7\pi}{4} & (c) & \quad \frac{7\pi}{6}
\end{align*}
\]

39. What is the period of the function 
\( g(t) = \sin 2t + \sin \frac{\theta}{2} \) ?

40. Assume that \( \sin \theta = \frac{4}{5} \), where \( \pi/2 < \theta < \pi \). Find:
\[
\begin{align*}
(a) & \quad \tan \theta & (b) & \quad \sin 2\theta & (c) & \quad \csc \frac{\theta}{2}
\end{align*}
\]

41. Give an example of values \( a, b \) such that
\[
\begin{align*}
(a) & \quad \cos(a + b) \neq \cos a + \cos b & (b) & \quad \cos \frac{a}{2} \neq \frac{\cos a}{2}
\end{align*}
\]

42. Let \( f(x) = \cos x \). Sketch the graph of 
\( y = 2f \left( \frac{1}{3}x - \frac{\pi}{4} \right) \) for \( 0 \leq x \leq 6\pi \).

43. Solve \( \sin 2x + \cos x = 0 \) for \( 0 \leq x < 2\pi \).

44. How does \( h(n) = n^2/2^n \) behave for large whole-number values of \( n \)? Does \( h(n) \) tend to infinity?

45. Use a graphing calculator to determine whether the equation \( \cos x = 5x^2 - 8x^4 \) has any solutions.

46. Using a graphing calculator, find the number of real roots and estimate the largest root to two decimal places:
\[
\begin{align*}
(a) f(x) = 1.8x^4 - x^5 - x & \quad (b) g(x) = 1.7x^4 - x^5 - x
\end{align*}
\]

47. Match each quantity (a)–(d) with (i), (ii), or (iii) if possible, or state that no match exists.
\[
\begin{align*}
(a) & \quad 2^a \ 3^b & (b) & \quad \frac{2a}{3b} & (c) & \quad (2^a)^b \\
(d) & \quad 2^{a-b} \ 3^{b-a} & (i) & \quad 2^{ab} & (ii) & \quad 6^{a+b} \\
(iii) & \quad \left( \frac{3}{2} \right)^{a-b}
\end{align*}
\]

48. Match each quantity (a)–(d) with (i), (ii), or (iii) if possible, or state that no match exists.
\[
\begin{align*}
(a) & \quad \ln \left( \frac{a}{b} \right) & (b) & \quad \frac{\ln a}{\ln b} & (c) & \quad e^{\ln a - \ln b} \\
(d) & \quad \ln a \ln b & (i) & \quad \ln a + \ln b & (ii) & \quad \ln a - \ln b \\
(iii) & \quad \frac{a}{b}
\end{align*}
\]

49. Find the inverse of \( f(x) = \sqrt{x^3 - 8} \) and determine its domain and range.

50. Find the inverse of \( f(x) = \frac{x - 2}{x - 1} \) and determine its domain and range.

51. Find a domain on which \( h(t) = (t - 3)^2 \) is one-to-one and determine the inverse on this domain.

52. Show that \( g(x) = \frac{x}{x - \ln x} \) is equal to its inverse on the domain \( \{ x : x \neq 1 \} \).

53. Suppose that \( g(x) \) is the inverse of \( f(x) \). Match the functions (a)–(d) with their inverses (i)–(iv).
\[
\begin{align*}
(a) & \quad f(x) + 1 & (b) & \quad f(x + 1) & (c) & \quad 4f(x) \\
(d) & \quad f(4x) & (i) & \quad g(x)/4 & (ii) & \quad g(x) \\
(iii) & \quad g(x - 1) & (iv) & \quad g(x) - 1
\end{align*}
\]

54. Plot \( f(x) = xe^x \) and use the zoom feature to find two solutions of \( f(x) = 0.5 \).