4.1: Linear Approximation and Applications

In Exercises 1–6, use Eq. (1) to estimate \( \Delta f = f(x + \Delta x) - f(x) \).

1. \( f(x) = x^2 \)  
2. \( f(x) = x^4 \)  
3. \( f(x) = x^{-1} \)  
4. \( f(x) = \frac{1}{x + 1} \)  
5. \( f(x) = \sqrt{x + 6} \)  
6. \( f(x) = \tan \left( \frac{\pi x}{3} \right) \)

7. The cube root of 27 is 3. How much larger is the cube root of 27.2? Estimate using the Linear Approximation.

8. Estimate \( \ln(e^3 + 0.1) - \ln(e^3) \) using differentials.

In Exercises 9–12, use Eq. (1) to estimate \( \Delta f \). Use a calculator to compute both the error and the percentage error.

9. \( f(x) = \sqrt{1 + x}, a = 3, \Delta x = 0.2 \)

10. \( f(x) = 2x^2 - x, a = 5, \Delta x = -0.4 \)

11. \( f(x) = \frac{1}{1 + x^2}, a = 3, \Delta x = 0.5 \)

12. \( f(x) = \ln(x^2 + 1), a = 1, \Delta x = 0.1 \)

In Exercises 13–16, estimate \( \Delta y \) using differentials [Eq. (3)].

13. \( y = \cos x, a = \frac{\pi}{6}, \Delta x = 0.014 \)

14. \( y = \tan^2 x, a = \frac{\pi}{4}, \Delta x = -0.02 \)

15. \( y = \frac{10 - x^2}{2 + x^2}, a = 1, \Delta x = 0.01 \)

16. \( y = x^{1/3} e^{-x}, a = 1, \Delta x = 0.1 \)

In Exercises 17–24, estimate using the Linear Approximation and find the error using a calculator.

17. \( \sqrt{26} - \sqrt{25} \)

18. \( 16.5^{1/4} - 16^{1/4} \)

19. \( \frac{1}{\sqrt{101}} - \frac{1}{10} \)

20. \( \frac{1}{\sqrt{98}} - \frac{1}{10} \)

21. \( 9^{1/3} - 2 \)

22. \( \tan^{-1}(1.05) - \frac{\pi}{4} \)

23. \( e^{0.1} - 1 \)

24. \( \ln(0.97) \)

25. Estimate \( f(4.03) \) for \( f(x) \) as in Figure 8.

In Exercises 26–31, estimate using the Linear Approximation and find the error using a calculator.

26. At a certain moment, an object in linear motion has velocity 100 m/s. Estimate the distance traveled over the next quarter-second, and explain how this is an application of the Linear Approximation.

27. Which is larger: \( \sqrt{2.1} - \sqrt{2} \) or \( \sqrt{9.1} - \sqrt{9} \)? Explain using the Linear Approximation.

28. Estimate \( \sin 61^\circ - \sin 60^\circ \) using the Linear Approximation. Hint: Express \( \Delta \theta \) in radians.

29. Box office revenue at a multiplex cinema in Paris is \( R(p) = 3600p - 10p^3 \) euros per showing when the ticket price is \( p \) euros. Calculate \( R(p) \) for \( p = 9 \) and use the Linear Approximation to estimate \( \Delta R \) if \( p \) is raised or lowered by 0.5 euros.

30. The stopping distance for an automobile is \( F(s) = 1.1s + 0.054s^2 \) ft, where \( s \) is the speed in mph. Use the Linear Approximation to estimate the change in stopping distance per additional mph when \( s = 35 \) and when \( s = 55 \).

31. A thin silver wire has length \( L = 18 \) cm when the temperature is \( T = 30^\circ C \). Estimate \( \Delta L \) when \( T \) decreases to \( 25^\circ C \) if the coefficient of thermal expansion is \( k = 1.9 \times 10^{-5}^\circ C^{-1} \) (see Example 3).
32. At a certain moment, the temperature in a snake cage satisfies \( \frac{dT}{dt} = 0.008 \text{°C/s} \). Estimate the rise in temperature over the next 10 seconds.

33. The atmospheric pressure at altitude \( h \) (kilometers) for \( 11 \leq h \leq 25 \) is approximately \( P(h) = 128e^{-0.157h} \) kilopascals.

(a) Estimate \( \Delta P \) at \( h = 20 \) when \( \Delta h = 0.5 \).

(b) Compute the actual change, and compute the percentage error in the Linear Approximation.

34. The resistance \( R \) of a copper wire at temperature \( T = 20 \text{°C} \) is \( R = 15 \Omega \). Estimate the resistance at \( T = 22 \text{°C} \), assuming that \( \frac{dR}{dT} \bigg|_{T=20} = 0.06 \Omega/°C \).

35. Newton’s Law of Gravitation shows that if a person weighs \( w \) pounds on the surface of the earth, then his or her weight at distance \( x \) from the center of the earth is

\[
W(x) = \frac{wR^2}{x^2} \quad (\text{for } x \geq R)
\]

where \( R = 3,960 \) miles is the radius of the earth (Figure 9).

(a) Show that the weight lost at altitude \( h \) miles above the earth’s surface is approximately \( \Delta W \approx -(0.0005w)h. \) \text{hint: use the Lin Approx: } \Delta x = h.

(b) Estimate the weight lost by a 200-lb football player flying in a jet at an altitude of 7 miles.

![FIGURE 9](image)

37. A stone tossed vertically into the air with initial velocity \( v \text{ cm/s} \) reaches a maximum height of \( h = \frac{v^2}{1960} \text{ cm} \).

(a) Estimate \( \Delta h \) if \( v = 700 \text{ cm/s} \) and \( \Delta v = 1 \text{ cm/s} \).

(b) Estimate \( \Delta h \) if \( v = 1,000 \text{ cm/s} \) and \( \Delta v = 1 \text{ cm/s} \).

(c) In general, does a 1 cm/s increase in \( v \) lead to a greater change in \( h \) at low or high initial velocities? Explain.

38. The side \( s \) of a square carpet is measured at 6 m. Estimate the maximum error in the area \( A \) of the carpet if \( s \) is accurate to within 2 centimeters.

39. A player located 18.1 ft from the basket launches a successful jump shot from a height of 10 ft (level with the rim of the basket), at an angle \( \theta = 34° \) and initial velocity \( v = 25 \text{ ft/s} \).

(a) Show that \( \Delta s \approx 0.255\Delta \theta \) ft for a small change of \( \Delta \theta \).

(b) Is it likely that the shot would have been successful if the angle had been off by 2°?

![FIGURE 10](image)

40. Estimate \( \Delta s \) if \( \theta = 34° \), \( v = 25 \text{ ft/s} \), and \( \Delta v = 2 \).
41. The radius of a spherical ball is measured at \( r = 25 \) cm. Estimate the maximum error in the volume and surface area if \( r \) is accurate to within 0.5 cm.

42. The dosage \( D \) of diphenhydramine for a dog of body mass \( w \) kg is \( D = 4.7w^{2/3} \) mg. Estimate the maximum allowable error in \( w \) for a cocker spaniel of mass \( w = 10 \) kg if the percentage error in \( D \) must be less than 3%.

43. The volume (in liters) and pressure \( P \) (in atmospheres) of a certain gas satisfy \( PV = 24 \). A measurement yields \( V = 4 \) with a possible error of \( \pm 0.3 \) L. Compute \( P \) and estimate the maximum error in this computation.

44. In the notation of Exercise 43, assume that a measurement yields \( V = 4 \). Estimate the maximum allowable error in \( V \) if \( P \) must have an error of less than 0.2 atm.

In Exercises 45–54, find the linearization at \( x = a \).

45. \( f(x) = x^4, a = 1 \)  
46. \( f(x) = \frac{1}{x}, a = 2 \)

47. \( f(x) = \sin^2 \theta, \ a = \frac{\pi}{4} \)  
48. \( f(x) = \frac{x^2}{x - 3}, a = 4 \)

49. \( y = (1 + x)^{-1/2}, a = 0 \)  
50. \( y = (1 + x)^{-1/2}, a = 3 \)

51. \( y = (1 + x^2)^{-1/2}, a = 0 \)  
52. \( y = \tan^{-1}x, a = 1 \)

53. \( y = e^{\sqrt{x}}, a = 1 \)  
54. \( y = e^x \ln x, a = 1 \)

55. What is \( f(2) \) if the linearization of \( f(x) \) at \( a = 2 \) is \( L(x) = 2x + 4 \)?

56. Compute the linearization of \( f(x) = 3x - 4 \) at \( a = 0 \) and \( a = 2 \). Prove more generally that a linear function coincides with its linearization at \( x = a \) for all \( a \).

57. Estimate \( \sqrt{16.2} \) using the linearization \( L(x) \) of \( f(x) = \sqrt{x} \) at \( a = 16 \). Plot \( f(x) \) and \( L(x) \) on the same set of axes and determine whether the estimate is too large or too small.

58. Estimate \( \frac{1}{\sqrt{15}} \) using a suitable linearization of \( f(x) = \frac{1}{\sqrt{x}} \). Plot \( f(x) \) and \( L(x) \) on the same set of axes and determine whether the estimate is too large or too small. Use a calculator to compute the percentage error.

In Exercises 59–67, approximate using linearization and use a calculator to compute the percentage error.

59. \( \frac{1}{\sqrt{17}} \)  
60. \( \frac{1}{101} \)  
61. \( \frac{1}{(10.03)^2} \)

62. \( \frac{1}{(17)^{1/4}} \)  
63. \( \frac{1}{(64.1)^{1/3}} \)  
64. \( \frac{1}{(1.2)^{53}} \)

65. \( \cos^{-1}(0.52) \)  
66. \( \ln 1.07 \)  
67. \( e^{-0.012} \)

68. Compute the linearization \( L(x) \) of \( f(x) = x^2 - x^3 \) at \( a = 4 \). Then plot \( f(x) - L(x) \) and find an interval \( I \) around \( a = 4 \) such that \( |f(x) - L(x)| \leq 0.1 \) for \( x \in I \).

69. Show that the Linear Approximation to \( f(x) = \sqrt{x} \) at \( x = 9 \) yields the estimate \( \sqrt{9 + h} - 3 \approx \frac{1}{6}h \). Set \( K = 0.001 \) and show that \( |f'(x)| \leq K \) for \( x \geq 9 \). Then verify numerically that the error \( E \) satisfies Eq. (5) for \( h = 10^{-n} \), for \( n \leq 4 \).

70. The Linear Approximation to \( f(x) = \tan x \) at \( x = \frac{\pi}{4} \) yields the estimate \( \tan (\frac{\pi}{4} + h) - 1 \approx 2h \). Set \( K = 6.2 \) and show, using a plot, that \( |f''(x)| \leq K \) for \( x \in [\frac{\pi}{4}, \frac{\pi}{4} + 0.1] \). Then verify numerically that the error \( E \) satisfies Eq. (5) for \( h = 10^{-n} \), for \( n \leq 4 \).

71. Compute \( dy/dx \) at the point \( P = (2, 1) \) on the curve \( y^3 + 3xy = 7 \) and show that the linearization at \( P \) is \( L(x) = -\frac{1}{3}x + \frac{5}{3} \). Use \( L(x) \) to estimate the \( y \)-coord of the point on the curve where \( x = 2.1 \).

72. Apply the method of Exercise 71 to \( P = (0, 5, 1) \) on \( y^3 + y - 2x = 1 \) to estimate the \( y \)-coordinate of the point on the curve where \( x = 0.55 \).
73. Apply the method of Exercise 71 to \( P = (-1, 2) \) on \( y^4 + 7xy = 2 \) to estimate the solution of \( y^4 - 7.7y = 2 \) near \( y = 2 \).

74. Show that for any real number \( k \), \((1 + \Delta x)^k \approx 1 + k\Delta x \) for small \( \Delta x \). Estimate \((1.02)^{0.7}\) and \((1.02)^{-0.3}\).

75. Let \( \Delta f = f(5 + h) - f(5) \), where \( f(x) = x^2 \). Verify directly that \( E = |\Delta f - f'(5)h| \) satisfies (5) with \( K = 2 \).

76. Let \( \Delta f = f(1 + h) - f(1) \) where \( f(x) = x^{-1} \). Show directly that \( E = |\Delta f - f'(1)h| \) is equal to \( h^2/(1 + h) \). Then prove that \( E \leq 2h^2 \) if \( -\frac{1}{2} \leq h \leq \frac{1}{2} \).

**4.2: Extreme Values**

1. The following questions refer to Figure 15.

   (a) How many critical pts does \( f(x) \) have on \([0, 8]\)?

   (b) What is the maximum value of \( f(x) \) on \([0, 8]\)?

   (c) What are the local maximum values of \( f(x) \)?

   (d) Find a closed interval on which both the minimum and maximum values of \( f(x) \) occur at critical points.

   (e) Find an interval on which the minimum value occurs at an endpoint.

2. State whether \( f(x) = x^{-1} \) (Figure 16) has a minimum or maximum value on the following intervals:

   (a) \((0, 2)\) \hspace{1cm} (b) \((1, 2)\) \hspace{1cm} (c) \([1, 2]\)

   ![Figure 16](attachment://fig16.png)

   In Exercises 3–20, find all critical points of the function.

3. \( f(x) = x^2 - 2x + 4 \) \hspace{1cm} 4. \( f(x) = 7x - 2 \)

5. \( f(x) = x^3 - \frac{9}{2}x^2 - 54x + 2 \) \hspace{1cm} 6. \( f(t) = 8t^3 - t^2 \)

7. \( f(x) = x^{-1} - x^{-2} \) \hspace{1cm} 8. \( g(z) = \frac{1}{z - 1} - \frac{1}{z} \)

9. \( f(x) = \frac{x}{x^2 + 1} \) \hspace{1cm} 10. \( f(x) = \frac{x^2}{x^2 - 4x + 8} \)

11. \( f(t) = t - 4\sqrt{t + 1} \) \hspace{1cm} 12. \( f(t) = 4t - \sqrt{t^2 + 1} \)

13. \( f(x) = x^2\sqrt{1 - x^2} \) \hspace{1cm} 14. \( f(x) = x + |2x + 1| \)

15. \( g(\theta) = \sin^2 \theta \) \hspace{1cm} 16. \( R(\theta) = \cos \theta + \sin^2 \theta \)

17. \( f(x) = x \ln x \) \hspace{1cm} 18. \( f(x) = x e^{2x} \)

19. \( f(x) = \sin^{-1} x - 2x \) \hspace{1cm} 20. \( f(x) = \sec^{-1} x - \ln x \)

21. Let \( f(x) = x^2 - 4x + 1 \).

   (a) Find the critical pt \( c \) of \( f(x) \) and compute \( f(c) \).

   (b) Compute the value of \( f(x) \) at the endpoints of the interval \([0, 4]\).

   (c) Determine the min and max of \( f(x) \) on \([0, 4]\).

   (d) Find the extreme values of \( f(x) \) on \([0, 1]\).

22. Find the extreme values of \( f(x) = 2x^3 - 9x^2 + 12x \) on \([0, 3]\) and \([0, 2]\).
23. Find the critical points of \( f(x) = \sin x + \cos x \) and determine the extreme values on \([0, \frac{\pi}{2}]\).

24. Compute the critical points of \( h(t) = (t^2 - 1)^{\frac{1}{3}} \). Check that your answer is consistent with Figure 17. Then find the extreme values of \( h(t) \) on \([0, 1]\) and \([0, 2]\).

25. Plot \( f(x) = 2\sqrt{x} - x \) on \([0, 4]\) and determine the maximum value graphically. Then verify your answer using calculus.

26. Plot \( f(x) = \ln x - 5 \sin x \) on \([0.1, 2]\) and approximate both the critical points and the extreme values.

27. Approximate the critical points of \( g(x) = x \cos^{-1} x \) and estimate the maximum value of \( g(x) \).

28. Approximate the crit pts of \( g(x) = 5e^x - \tan x \) in \((-\frac{\pi}{2}, \frac{\pi}{2})\).

In Exercises 29–58, find the min and max of the function on the given interval by comparing values at the critical points and endpoints.

29. \( y = 2x^2 + 4x + 5, [-2, 2] \)
30. \( y = 2x^2 + 4x + 5, [0, 2] \)
31. \( y = 6t - t^2, [0, 5] \)
32. \( y = 6t - t^2, [4, 6] \)
33. \( y = x^3 - 6x^2 + 8, [1, 6] \)
34. \( y = x^3 + x^2 - x, [-2, 2] \)
35. \( y = 2t^3 + 3t^2, [1, 2] \)
36. \( y = x^3 - 12x^2 + 21x, [0, 2] \)
37. \( y = z^5 - 80z, [-3, 3] \)
38. \( y = 2x^5 + 5x^2, [-2, 2] \)
39. \( y = \frac{x^2 + 1}{x - 4}, [5, 6] \)
40. \( y = \frac{1-x}{x^2 + 3x}, [1, 4] \)
41. \( y = x - \frac{4x}{x+1}, [0, 3] \)
42. \( y = 2\sqrt{x^2 + 1} - x, [0, 2] \)
43. \( y = (2 + x)\sqrt{2 + (2 - x)^2}, [0, 2] \)
44. \( y = \sqrt{1 + x^2 - 2x}, [0, 1] \)
45. \( y = \sqrt{x + x^2 - 2\sqrt{x}}, [0, 4] \)
46. \( y = (t - r)^{1/3}, [-1, 2] \)
47. \( y = \sin x \cos x, [0, \frac{\pi}{2}] \)
48. \( y = x + \sin x, [0, 2\pi] \)
49. \( y = \sqrt{2} \theta - \sec \theta, [0, \frac{\pi}{3}] \)
50. \( y = \cos \theta + \sin \theta, [0, 2\pi] \)
51. \( y = \theta - 2 \sin \theta, [0, 2\pi] \)
52. \( y = 4 \sin^3 \theta - 3 \cos^2 \theta, [0, 2\pi] \)
53. \( y = \tan x - 2x, [0, 1] \)
54. \( y = xe^{-x}, [0, 2] \)
55. \( y = \frac{\ln x}{x}, [1, 3] \)
56. \( y = 3e^x - e^{2x}, [-\frac{1}{2}, 1] \)
57. \( y = 5 \tan^{-1} x - x, [1, 5] \)
58. \( y = x^3 - 24 \ln x, [\frac{1}{2}, 3] \)
59. Let \( f(\theta) = 2 \sin 2\theta + \sin 40. \)

(a) Show that \( \theta \) is a critical pt if \( \cos 4\theta = -\cos 2\theta \).

(b) Show, using a unit circle, that \( \cos \theta_1 = -\cos \theta_2 \) if and only if \( \theta_1 = \pi + \theta_2 + 2\pi k \) for an integer \( k \).

(c) Show that \( \cos 4\theta = -\cos 2\theta \) if and only if \( \theta = \frac{\pi}{2} + \pi k \) or \( \theta = \frac{\pi}{6} + (\frac{\pi}{3})k \).

(d) Find the six critical points of \( f(\theta) \) on \([0, 2\pi]\)
and find the extreme values of \( f(\theta) \) on this interval.

(e) Check your results against a graph of \( f(\theta) \).
60. Find the critical points of \( f(x) = 2 \cos 3x + 3 \cos 2x \) in \([0, 2\pi]\). Check your answer against a graph of \( f(x) \).

In Exercises 61–64, find the critical points and the extreme values on \([0, 2\pi]\). In Exercise 63 and 64, refer to Figure 18.

61. \( y = |x - 2| \)
62. \( y = |3x - 9| \)
63. \( y = |x^2 + 4x - 12| \)
64. \( y = |\cos x| \)

![Graph of \( y = |x^2 + 4x - 12| \) and \( y = |\cos x| \)](image)

**FIGURE 18**

In Exercises 65–68, verify Rolle’s Theorem for the given interval.

65. \( f(x) = x + x^{-1}, \left[\frac{1}{2}, 2\right] \)
66. \( f(x) = \sin x, \left[\frac{\pi}{4}, \frac{3\pi}{4}\right] \)
67. \( f(x) = \frac{x^2}{8x - 15}, [3, 5] \)
68. \( f(x) = \sin^2 x - \cos^2 x, \left[\frac{\pi}{4}, \frac{3\pi}{4}\right] \)

69. Prove that \( f(x) = x^5 + 4x^3 + 4x - 12 \) has precisely one real root.

70. Prove that \( f(x) = x^3 + 3x^2 + 6x \) has precisely one real root.

71. Prove that \( f(x) = x^4 + 5x^3 + 4x \) has no root \( c \) satisfying \( c > 0 \). **Hint**: Note that \( x = 0 \) is a root and apply Rolle’s Theorem.

72. Prove that \( c = 4 \) is the largest root of \( f(x) = x^4 - 8x^2 - 128 \).

73. The position of a mass oscillating at the end of a spring is \( s(t) = A \sin \omega t \), where \( A \) is the amplitude and \( \omega \) is the angular frequency. Show that the speed \( |v(t)| \) is at a maximum when the acceleration \( a(t) \) is zero and that \( |a(t)| \) is at a maximum when \( v(t) \) is zero.

74. The concentration \( C(t) \) (in mg/cm\(^3\)) of a drug in a patient’s bloodstream after \( t \) hours is

\[
C(t) = \frac{0.016t}{t^2 + 4t + 4}
\]

Find the maximum concentration in the time interval \([0, 8]\) and the time at which it occurs.

75. **Antibiotic Levels** A study shows that the concentration \( C(t) \) (in micrograms per milliliter) of antibiotic in a patient’s blood serum after \( t \) hours is

\[
C(t) = 120(e^{-0.2t} - e^{-bt})
\]

where \( b \geq 1 \) is a constant that depends on the particular combination of antibiotic agents used. Solve numerically for the value of \( b \) (to two decimal places) for which maximum concentration occurs at \( t = 1 \) h. You may assume that the maximum occurs at a critical point as suggested by Fig 19.

![Graph of \( C(t) = 120(e^{-0.2t} - e^{-bt}) \) with \( b \) chosen so that the maximum occurs at \( t = 1 \) h.](image)

**FIGURE 19**

76. In the notation of Exercise 75, find the value of \( b \) (to two decimal places) for which the maximum value of \( C(t) \) is equal to 100 mcg/ml.

(a) Show that \( \theta \approx 54.7^\circ \) (assume \( h \) and \( s \) are constant). **Hint**: Find the critical point of \( A(\theta) \) for \( 0 < \theta < \pi/2 \).

(b) Confirm, by graphing \( f(\theta) = \sqrt{3} \csc \theta - \cot \theta \), that the critical point indeed minimizes the surface area.
81. Find the maximum of \( y = x^a - x^b \) on [0, 1] where \( 0 < a < b \). In particular, find the maximum of \( y = x^5 - x^{10} \) on [0, 1].

In Exercises 82–84, plot the function using a graphing utility and find its critical points and extreme values on \([-5, 5]\).

\[
y = \frac{1}{1 + |x - 1|}
\]

82.

\[
y = \frac{1}{1 + |x - 1|} + \frac{1}{1 + |x - 4|}
\]

83.

\[
y = \frac{x}{|x^2 - 1| + |x^2 - 4|}
\]

84.

85. (a) Use implicit differentiation to find the critical points on the curve \( 27x^2 = (x^2 + y^2)^3 \).

(b) Plot the curve and the horizontal tangent lines on the same set of axes.

86. Sketch the graph of a continuous function on (0, 4) with a minimum value but no maximum value.

87. Sketch the graph of a continuous function on (0, 4) having a local minimum but no absolute minimum.

88. Sketch the graph of a function on [0, 4] having (a) Two local maxima and one local minimum.

(b) An absolute minimum that occurs at an endpoint, and an absolute maximum that occurs at a critical point.

89. Sketch the graph of a function \( f(x) \) on [0, 4] with a discontinuity such that \( f(x) \) has an absolute minimum but no absolute maximum.

90. A rainbow is produced by light rays that enter a raindrop (assumed spherical) and exit after being reflected internally as in Figure 24. The angle between the incoming and reflected rays is \( \theta = 4r - 2i \), where the angle of incidence \( i \) and refraction \( r \) are related by Snell’s Law \( \sin i = n \sin r \) with \( n \approx 1.33 \) (the index of refraction for air and water).

(a) Use Snell’s Law to show that

\[
\frac{dr}{di} = \frac{\cos i}{n \cos r}
\]

(b) Show that the maximum value \( \theta_{\text{max}} \) of \( \theta \) occurs when \( i \) satisfies

\[
\cos i = \sqrt{\frac{n^2 - 1}{3}}. \quad \text{Hint: Show that} \quad \frac{d\theta}{di} = 0 \quad \text{if} \quad \cos i = \frac{n}{2 \cos r}. \quad \text{Then use Snell’s Law to eliminate} \ r.
\]

(c) Show that \( \theta_{\text{max}} \approx 59.58^\circ \).

91. Show that the extreme values of \( f(x) = a \sin x + b \cos x \) are \( \pm \sqrt{a^2 + b^2} \).

92. Show, by considering its minimum, that \( f(x) = x^2 - 2x + 3 \) takes on only positive values. More generally, find the conditions on \( r \) and \( s \) under which the quadratic function \( f(x) = x^2 + rx + s \) takes on only positive values. Give examples of \( r \) and \( s \) for which \( f \) takes on both positive and negative values.

93. Show that if the quadratic polynomial \( f(x) = x^2 + rx + s \) takes on both positive and negative values, then its minimum value occurs at the midpoint between the two roots.

94. Generalize Exercise 93: Show that if the horizontal line \( y = c \) intersects the graph of \( f(x) = \)
\[ x^2 + rx + s \] at two points \((x_1, f(x_1))\) and \((x_2, f(x_2))\), then \(f(x)\) takes its minimum value at the midpoint \(M = \frac{x_1 + x_2}{2}\) (Figure 25).

95. A cubic polynomial may have a local min and max, or it may have neither (Figure 26). Find conditions on the coefficients \(a\) and \(b\) of that ensure that \(f\) has neither a local min nor a local max. Hint: Apply Exercise 92 to \(f(x)\).

96. Find the min and max of
\[
 f(x) = \frac{1}{3}x^3 + \frac{1}{2}ax^2 + bx + c
\]

97. Prove that if \(f\) is continuous and \(f(a)\) and \(f(b)\) are local minima where \(a < b\), then there exists a value \(c\) between \(a\) and \(b\) such that \(f(c)\) is a local maximum. (Hint: Apply Theorem 1 to the interval \([a, b]\).) Show that continuity is a necessary hypothesis by sketching the graph of a function (necessarily discontinuous) with two local minima but no local maximum.

**4.3: MVT and Monotonicity**

In Exercises 1–8, find a point \(c\) satisfying the conclusion of the MVT for the given function and interval.

1. \(y = x^{-1}, [2, 8]\)  
2. \(y = \sqrt{x}, [9, 25]\)  
3. \(y = \cos x - \sin x, [0, 2\pi]\)  
4. \(y = \frac{x}{x + 2}, [1, 4]\)  
5. \(y = x^3, [-4, 5]\)  
6. \(y = x \ln x, [1, 2]\)  
7. \(y = e^{-2x}, [0, 3]\)  
8. \(y = e^x - x, [-1, 1]\)  
9. Let \(f(x) = x^5 + x^2\). The secant line between \(x = 0\) and \(x = 1\) has slope 2 (check this), so by the MVT, \(f(c) = 2\) for some \(c \in (0, 1)\). Plot \(f(x)\) and the secant line on the same axes. Then plot \(y = 2x + b\) for different values of \(b\) until the line becomes tangent to the graph of \(f\). Zoom in on the point of tangency to estimate \(x\)-coordinate \(c\) of the point of tangency.

10. Plot the derivative of \(f(x) = 3x^5 - 5x^3\). Describe its sign changes and use this to determine the local extreme values of \(f(x)\). Then graph \(f(x)\) to confirm your conclusions.

11. Determine the intervals on which \(f(x)\) is positive and negative, assuming that Figure 13 is the graph of \(f(x)\).

12. Determine the intervals on which \(f(x)\) is increasing or decreasing, assuming that Figure 13 is the graph of \(f(x)\).

13. State whether \(f(2)\) and \(f(4)\) are local minima or local maxima, assuming that Figure 13 is the graph of \(f(x)\).

14. Figure 14 shows the graph of the derivative \(f'(x)\) of a function \(f(x)\). Find the critical points of \(f(x)\) and determine whether they are local minima, local maxima, or neither.
In Exercises 15–18, sketch the graph of a function $f(x)$ whose derivative $f'(x)$ has the given description.

15. $f'(x) > 0$ for $x > 3$ and $f'(x) < 0$ for $x < 3$
16. $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$
17. $f'(x)$ is negative on $(1, 3)$ and positive everywhere else.
18. $f'(x)$ makes the sign transitions $+,-,+,-$.

In Exercises 19–22, find all critical points of $f$ and use the First Derivative Test to determine whether they are local minima or maxima.

19. $f(x) = 4 + 6x - x^2$  
20. $f(x) = x^3 - 12x - 4$
21. $f(x) = \frac{x^2}{x + 1}$  
22. $f(x) = x^3 + x^3$

In Exercises 23–52, find the critical points and the intervals on which the function is increasing or decreasing. Use the First Derivative Test to determine whether the critical point is a local min or max (or neither).

23. $y = -x^2 + 7x - 17$  
24. $y = 5x^2 + 6x - 4$
25. $y = x^3 - 12x^2$  
26. $y = x(x - 2)^3$
27. $y = 3x^4 + 8x^3 - 6x^2 - 24x$
28. $y = x^2 + (10 - x)^2$
29. $y = \frac{1}{3}x^3 + \frac{3}{4}x^2 + 2x + 4$
30. $y = x^4 + x^3$  
31. $y = x^5 + x^3 + 1$
32. $y = x^5 + x^3 + x$  
33. $y = x^4 - 4x^{3/2}$ $(x > 0)$
34. $y = x^{5/2} - x^2$ $(x > 0)$  
35. $y = x + x^{-1}$ $(x > 0)$
36. $y = x^{-2} - 4x^{-1}$ $(x > 0)$  
37. $y = \frac{1}{x^2 + 1}$
38. $y = \frac{2x + 1}{x^2 + 1}$  
39. $y = \frac{x^3}{x^2 + 1}$
40. $y = \frac{x^3}{x^2 - 3}$  
41. $y = \theta + \sin \theta + \cos \theta$
42. $y = \sin \theta + \sqrt{3} \cos \theta$  
43. $y = \sin^2 \theta + \sin \theta$
44. $y = \theta - 2 \cos \theta$, $[0, 2\pi]$  
45. $y = x + e^{-x}$
46. $y = \frac{e^x}{x}$ $(x > 0)$  
47. $y = e^{-\cos x}$, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
48. $y = x^2 e^x$  
49. $y = \tan^{-1} x - \frac{1}{2} x$
50. $y = (x^2 - 2x)e^x$  
51. $y = x - \ln x$ $(x > 0)$
52. $y = \frac{\ln x}{x}$ $(x > 0)$

53. Find the minimum value of $f(x) = x^2$ for $x > 0$.
54. Show that $f(x) = x^2 + bx + c$ is decreasing on $(-\infty, -\frac{b}{2})$ and increasing on $(-\frac{b}{2}, \infty)$
55. Show that $f(x) = x^3 - 2x^2 + 2x$ is an increasing function. Hint: Find the minimum value of $f'(x)$.

56. Find conditions on $a$ and $b$ that ensure that $f(x) = x^3 + ax + b$ is increasing on $(-\infty, \infty)$.

57. Let $h(x) = \frac{x(x^2 - 1)}{x^2 + 1}$ and suppose that $f'(x) = h(x)$. Plot $h(x)$ and use the plot to describe the local extrema and the increasing/decreasing behavior of $f(x)$. Sketch a plausible graph for $f(x)$.

58. Sam made two statements that Deborah found dubious.

(a) “The average velocity for my trip was 70 mph; at no point in time did my speedometer read 70 mph.”
(b) “A policeman clocked me going 70 mph, but my speedometer never read 65 mph.”

In each case, which theorem did Deborah apply to prove Sam’s statement false: the Intermediate Value Theorem or the Mean Value Theorem? Explain.

59. Determine where \( f(x) = (1,000 - x)^2 + x^2 \) is decreasing. Use this to decide which is larger: \( 800^2 + 200^2 \) or \( 600^2 + 400^2 \).

60. Show that \( f(x) = 1 - |x| \) satisfies the conclusion of the MVT on \([a, b]\) if both \( a \) and \( b \) are positive or negative, but not if \( a < 0 \) and \( b > 0 \).

61. Which values of \( c \) satisfy the conclusion of the MVT on the interval \([a, b]\) if \( f(x) \) is a linear function?

62. Show that if \( f(x) \) is any quadratic polynomial, then the midpoint \( c = \frac{a + b}{2} \) satisfies the conclusion of the MVT on \([a, b]\) for any \( a \) and \( b \).

63. Suppose that \( f(0) = 2 \) and \( f'(x) \leq 3 \) for \( x > 0 \). Apply the MVT to the interval \([0, 4]\) to prove that \( f(4) \leq 14 \). Prove more generally that \( f(x) \leq 2 + 3x \) for all \( x > 0 \).

64. Show that if \( f(2) = -2 \) and \( f'(x) \geq 5 \) for \( x > 2 \), then \( f(4) \geq 8 \).

65. Show that if \( f(2) = 5 \) and \( f'(x) \geq 10 \) for \( x > 2 \), then \( f(x) \geq 10x - 15 \) for all \( x > 2 \).

66. Show that a cubic function \( f(x) = x^3 + ax^2 + bx + c \) is increasing on \((-\infty, \infty)\) if \( b > a^2/3 \).

67. Prove that if \( f(0) = g(0) \) and \( f'(x) \leq g'(x) \) for \( x \geq 0 \), then \( f(x) \leq g(x) \) for all \( x \geq 0 \). Hint: Show that \( f'(x) - g'(x) \) is nonincreasing.

68. Use Exercise 67 to prove that \( x \leq \tan x \) for \( 0 \leq x < \pi/2 \).

69. Use Exercise 67 and the inequality \( \sin x \leq x \) for \( x \geq 0 \) (established in Theorem 3 of Section 2.6) to prove the following assertions for all \( x \geq 0 \) (each assertion follows from the previous one).

(a) \( \cos x \geq 1 - \frac{1}{2}x^2 \)

(b) \( \sin x \geq x - \frac{1}{6}x^3 \)

(c) \( \cos x \leq 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \)

(d) Can you guess the next inequality in the series?

70. Let \( f(x) = e^{-x} \). Use the method of Exercise 69 to prove the following inequalities for \( x \geq 0 \).

(a) \( e^{-x} \geq 1 - x \)

(b) \( e^{-x} \leq 1 - x + \frac{1}{2}x^2 \)

(c) \( e^{-x} \geq 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 \)

Can you guess the next inequality in the series?

71. Assume that \( f'' \) exists and \( f''(x) = 0 \) for all \( x \). Prove that \( f(x) = mx + b \), where \( m = f'(0) \) and \( b = f(0) \).

72. Define \( f(x) = x^3 \sin \left(\frac{1}{x}\right) \) for \( x \neq 0 \) and \( f(0) = 0 \).

(a) Show that \( f'(x) \) is continuous at \( x = 0 \) and that \( x = 0 \) is a critical point of \( f \).

(b) Examine the graphs of \( f(x) \) and \( f'(x) \). Can the First Derivative Test be applied?

(c) Show that \( f(0) \) is neither a local min nor a local max.

73. Suppose that \( f(x) \) satisfies the following equation (an example of a differential equation):

\[
 f''(x) = -f(x) 
\]
(a) Show that \( f(x)^2 + f'(x)^2 = f(0)^2 + f(0)^2 \) for all \( x \). *Hint:* Show that the function on the left has zero derivative.

(b) Verify that \( \sin x \) and \( \cos x \) satisfy Eq. (1), and deduce that \( \sin^2 x + \cos^2 x = 1 \).

74. Suppose that functions \( f \) and \( g \) satisfy Eq. (1) and have the same initial values—that is, \( f(0) = g(0) \) and \( f'(0) = g'(0) \). Prove that \( f(x) = g(x) \) for all \( x \). *Hint:* Apply Exercise 73(a) to \( f - g \).

75. Use Exercise 74 to prove: \( f(x) = \sin x \) is the unique solution of Eq. (1) such that \( f(0) = 0 \) and \( f'(0) = 1 \); and \( g(x) = \cos x \) is the unique solution such that \( g(0) = 1 \) and \( g'(0) = 0 \). This result can be used to develop all the properties of the trigonometric functions “analytically”—that is, without reference to triangles.

### 4.4: The Shape of a Graph

1. Match the graphs in Figure 13 with the description:
   - (a) \( f'(x) < 0 \) for all \( x \).
   - (b) \( f'(x) \) goes from + to −.
   - (c) \( f'(x) > 0 \) for all \( x \).
   - (d) \( f'(x) \) goes from − to +.

![Figure 13](image)

2. Match each statement with a graph in Figure 14 that represents company profits as a function of time.

   - (a) The outlook is great: The growth rate keeps increasing.
   - (b) We’re losing money, but not as quickly as before.
   - (c) We’re losing money, and it’s getting worse as time goes on.
   - (d) We’re doing well, but our growth rate is leveling off.
   - (e) Business had been cooling off, but now it’s picking up.
   - (f) Business had been picking up, but now it’s cooling off.

![Figure 14](image)

In Exercises 3–18, determine the intervals on which the function is concave up or down and find the points of inflection.

3. \( y = x^2 - 4x + 3 \)  
4. \( y = t^3 - 6t^2 + 4 \)
5. \( y = 10x^3 - x^5 \)  
6. \( y = 5x^2 + x^4 \)
7. \( y = \theta - 2 \sin \theta \), \( [0, 2\pi] \)  
8. \( y = \theta + \sin^2 \theta \), \( [0, \pi] \)
9. \( y = x(x - 8\sqrt{x}) \), \( (x \geq 0) \)
10. \( y = x^{7/2} - 35x^2 \)  
11. \( y = (x-2)(1-x^3) \)
12. \( y = x^{7/5} \)  
13. \( y = \frac{1}{x^2 + 3} \)  
14. \( y = \frac{x}{x^2 + 9} \)
15. \( y = xe^{-3x} \)  
16. \( y = (x^2 - 7)e^x \)
17. \( y = 2x^2 + \ln x \) (\( x > 0 \))  
18. \( y = x - \ln x \) (\( x > 0 \))

19. The growth of a sunflower during the first 100 days after sprouting is modeled well by the logistic curve \( y = h(t) \) shown in Figure 15. Estimate the growth rate at the point of inflection and explain its significance. Then make a rough sketch of the first and second derivatives of \( h(t) \).
20. Assume that Figure 16 is the graph of \( f(x) \). Where do the points of inflection of \( f(x) \) occur, and on which interval is \( f(x) \) concave down?

21. Repeat Exercise 20 but assume that Figure 16 is the graph of the derivative \( f'(x) \).

22. Repeat Exercise 20 but assume that Figure 16 is the graph of the second derivative \( f''(x) \).

23. Figure 17 shows the derivative \( f'(x) \) on \([0, 1.2]\). Locate the points of inflection of \( f(x) \) and the points where the local minima and maxima occur. Determine the intervals on which \( f(x) \) has the following properties:

(a) Increasing  
(b) Decreasing  
(c) Concave up  
(d) Concave down

24. Leticia has been selling solar-powered laptop chargers through her website, with monthly sales as recorded below. In a report to investors, she states, “Sales reached a point of inflection when I started using pay-per-click advertising.” In which month did that occur? Explain.

<table>
<thead>
<tr>
<th>Month</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sales</td>
<td>2</td>
<td>30</td>
<td>50</td>
<td>60</td>
<td>90</td>
<td>150</td>
<td>230</td>
<td>340</td>
</tr>
</tbody>
</table>

In Exercises 25–38, find the critical points and apply the Second Derivative Test (or state that it fails).

25. \( f(x) = x^3 - 12x^2 + 45x \)  
26. \( f(x) = x^4 - 8x^2 + 1 \)  
27. \( f(x) = 3x^4 - 8x^3 + 6x^2 \)  
28. \( f(x) = x^5 - x^3 \)

29. \( f(x) = \frac{x^2 - 8x}{x + 1} \)  
30. \( f(x) = \frac{1}{x^2 - x + 2} \)

31. \( y = 6x^{3/2} - 4x^{1/2} \)  
32. \( y = 9x^{7/3} - 21x^{1/2} \)

33. \( f(x) = \sin^2 x + \cos x, \quad [0, \pi] \)

34. \( y = \frac{1}{\sin x + 4}, \quad [0, 2\pi] \)  
35. \( f(x) = xe^{-x^2} \)

36. \( f(x) = e^{-x} - 4e^{-2x} \)  
37. \( f(x) = x^3 \ln x \quad (x > 0) \)

38. \( f(x) = \ln x + \ln(4 - x^2), \quad (0, 2) \)

In Exercises 39–52, find the intervals on which \( f \) is concave up or down, the points of inflection, the critical points, and the local minima and maxima.

39. \( f(x) = x^3 - 2x^2 + x \)  
40. \( f(x) = x^2(x - 4) \)

41. \( f(t) = t^3 - t \)  
42. \( f(x) = 2x^4 - 3x^3 + 2 \)

43. \( f(x) = x^2 - 8x^{1/2} \quad (x \geq 0) \)

44. \( f(x) = x^{3/2} - 4x^{-1/2} \quad (x > 0) \)  
45. \( f(x) = \frac{x}{x^2 + 27} \)

46. \( f(x) = \frac{1}{x^4 + 1} \)  
47. \( f(\theta) = \theta + \sin \theta, \quad [0, 2\pi] \)

48. \( f(x) = \cos^2 x, \quad [0, \pi] \)  
49. \( f(x) = \tan x, \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \)

50. \( f(x) = e^{-x} \cos x, \quad \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \)

51. \( y = (x^2 - 2)e^{-x} \quad (x > 0) \)  
52. \( y = \ln(x^2 + 2x + 5) \)
53. Sketch the graph of an increasing function such that \( f''(x) \) changes from + to − at \( x = 2 \) and from − to + at \( x = 4 \). Do the same for a dec. function.

In Exercises 54–56, sketch the graph of a function \( f(x) \) satisfying all of the given conditions.

54. \( f'(x) > 0 \) and \( f''(x) < 0 \) for all \( x \).

55. (i) \( f'(x) > 0 \) for all \( x \), and
(ii) \( f''(x) < 0 \) for \( x < 0 \) and \( f''(x) > 0 \) for \( x > 0 \).

56. (i) \( f'(x) < 0 \) for \( x < 0 \) and \( f'(x) > 0 \) for \( x > 0 \), and
(ii) \( f''(x) < 0 \) for \( |x| > 2 \), and \( f''(x) > 0 \) for \( |x| < 2 \).

57. An infectious flu spreads slowly at the beginning of an epidemic. The infection process accelerates until a majority of the susceptible individuals are infected, at which point the process slows down.

(a) If \( R(t) \) is the number of individuals infected at time \( t \), describe the concavity of the graph of \( R \) near the beginning and end of the epidemic.

(b) Describe the status of the epidemic on the day that \( R(t) \) has a point of inflection.

58. Water is pumped into a sphere at a constant rate (Figure 18). Let \( h(t) \) be the water level at time \( t \). Sketch the graph of \( h(t) \) (approximately, but with the correct concavity). Where does the point of inflection occur?

59. Water is pumped into a sphere of radius \( R \) at a variable rate in such a way that the water level rises at a constant rate (Figure 18). Let \( V(t) \) be the volume of water in the tank at time \( t \). Sketch the graph \( V(t) \) (approximately, but with the correct concavity). Where does the pt of inflection occur?

60. (Continuation of Exercise 59) If the sphere has radius \( R \), the volume of water is \( V = \pi \left(Rh^2 - \frac{1}{3}h^3\right) \), where \( h \) is the water level. Assume the level rises at a constant rate of 1 (that is, \( h = t \)).

(a) Find the inflection point of \( V(t) \). Does this agree with your conclusion in Exercise 59?

(b) Plot \( V(t) \) for \( R = 1 \).

62. Use graphical reasoning to determine whether the following statements are true or false. If false, modify the statement to make it correct.

(a) If \( f(x) \) is increasing, then \( f^{-1}(x) \) is decreasing.

(b) If \( f(x) \) is decreasing, then \( f^{-1}(x) \) is decreasing.

(c) If \( f(x) \) is concave up, then \( f^{-1}(x) \) is concave up.

(d) If \( f(x) \) is concave down, then \( f^{-1}(x) \) is concave up.

In Exercises 63–65, assume that \( f(x) \) is differentiable.

63. Proof of the Second Derivative Test Let \( c \) be a critical point such that \( f''(c) > 0 \) (the case \( f''(c) < 0 \) is similar).

(a) Show that \[ f''(c) = \lim_{h \to 0} \frac{f'(c + h)}{h}. \]

(b) Use (a) to show that there exists an open interval \( (a, b) \) containing \( c \) such that \( f'(x) < 0 \) if \( a < x < c \) and \( f'(x) > 0 \) if \( c < x < b \). Conclude that \( f(c) \) is a local minimum.
64. Prove that if \( f''(x) \) exists and \( f''(x) > 0 \) for all \( x \), then the graph of \( f(x) \) “sits above” its tangent lines.

(a) For any \( c \), set \( G(x) = f(x) - f'(c)(x - c) - f(c) \). It is sufficient to prove that \( G(x) \geq 0 \) for all \( c \).

(b) Show that \( G(c) = G'(c) = 0 \) and \( G''(x) > 0 \) for all \( x > c \). Conclude that \( G(x) < 0 \) for \( x < c \) and \( G(x) > 0 \) for \( x > c \). Then deduce, using the MVT, that \( G(x) > G(c) \) for \( x \neq c \).

65. Assume that \( f''(x) \) exists and let \( c \) be a point of inflection of \( f(x) \).

(a) Use the method of Exercise 64 to prove that the tangent line at \( x = c \) crosses the graph (Figure 21). Hint: Show that \( G(x) \) changes sign at \( x = c \).

(b) Verify this conclusion for \( f(x) \) by graphing \( f(x) \) and the tangent line at each inflection point on the same set of axes.

66. Let \( C(x) \) be the cost of producing \( x \) units of a certain good. Assume that the graph of \( C(x) \) is concave up.

(a) Show that the average cost \( A(x) = C(x)/x \) is minimized at the production level \( x_0 \) such that average cost equals marginal cost—that is, \( A(x_0) = C'(x_0) \).

(b) Show that the line through \( (0, 0) \) and \( (x_0, C(x_0)) \) is tangent to the graph of \( C(x) \).

67. Let \( f(x) \) be a polynomial of degree \( n \geq 2 \). Show that \( f(x) \) has at least one point of inflection if \( n \) is odd. Then give an example to show that \( f(x) \) need not have a point of inflection if \( n \) is even.

68. Critical and Inflection Points If \( f'(c) = 0 \) and \( f(c) \) is neither a local min nor a local max, must \( x = c \) be a point of inflection? This is true for “reasonable” functions (including the functions studied in this text), but it is not true in general. Let

\[
 f(x) = \begin{cases} 
  x^2 \sin \frac{1}{x} & \text{for } x \neq 0 \\
  0 & \text{for } x = 0 
\end{cases}
\]

(a) Use the limit definition of the derivative to show that \( f'(0) \) exists and \( f'(0) = 0 \).

(b) Show that \( f(0) \) is neither a local min nor a local max.

(c) Show that \( f'(x) \) changes sign infinitely often near \( x = 0 \). Conclude that \( x = 0 \) is not a point of inflection.

4.5: L’Hospital’s Rule

In Exercises 1–10, use L’Hôpital’s Rule to evaluate the limit, or state that L’Hôpital’s Rule does not apply.

\[
\begin{align*}
1 & \lim_{x \to 3} \frac{2x^2 - 5x - 3}{x - 4} \\
2 & \lim_{x \to -5} \frac{x^2 - 25}{x + 5} \\
3 & \lim_{x \to 4} \frac{x^3 - 64}{x^2 + 16} \\
4 & \lim_{x \to -1} \frac{x^4 + 2x + 1}{x^5 - 2x - 1} \\
5 & \lim_{x \to 9} \frac{x^{1/2} + x - 6}{x^{3/2} - 27} \\
6 & \lim_{x \to 3} \frac{\sqrt{x + 1} - 2}{x^3 - 7x - 6} \\
7 & \lim_{x \to 0} \frac{\sin 4x}{x^2 + 3x + 1} \\
8 & \lim_{x \to 0} \frac{x^3}{\sin x - x} \\
9 & \lim_{x \to 0} \frac{\cos 2x - 1}{\sin 5x} \\
10 & \lim_{x \to 0} \frac{\cos x - \sin^2 x}{\sin x}
\end{align*}
\]
In Exercises 11–16, show that L'Hôpital's Rule is applicable to the limit as $x \to \pm\infty$ and evaluate.

11. $\lim_{x \to \infty} \frac{9x + 4}{3 - 2x}$
12. $\lim_{x \to \infty} x \sin \frac{1}{x}$
13. $\lim_{x \to \infty} \frac{\ln x}{x^{1/2}}$
14. $\lim_{x \to \infty} \frac{x}{e^x}$
15. $\lim_{x \to -\infty} \frac{\ln(x^4 + 1)}{x}$
16. $\lim_{x \to \infty} \frac{x^2}{e^x}$

In Exercises 17–54, evaluate the limit.

17. $\lim_{x \to 1} \frac{\sqrt{8 + x} - 3x^{1/3}}{x^2 - 3x + 2}$
18. $\lim_{x \to 4} \left[ \frac{1}{\sqrt{x - 2}} - \frac{4}{x - 4} \right]$
19. $\lim_{x \to \infty} \frac{3x - 2}{1 - 5x}$
20. $\lim_{x \to \infty} \frac{x^{2/3} + 3x}{x^{5/3} - x}$
21. $\lim_{x \to \infty} \frac{7x^2 + 4x}{9 - 3x^2}$
22. $\lim_{x \to \infty} \frac{3x^3 + 4x^2}{4x^3 - 7}$
23. $\lim_{x \to 1} (1 + 3x)^{1/2} - 2$
24. $\lim_{x \to 1} (1 + 7x)^{1/3} - 2$
25. $\lim_{x \to 0} \frac{x^{5/3} - 2x - 16}{x^{1/3} - 2}$
26. $\lim_{x \to 0} \frac{\tan 4x}{\tan 5x}$
27. $\lim_{x \to 0} \frac{\tan x}{x}$
28. $\lim_{x \to 0} \left( \cot x - \frac{1}{x} \right)$
29. $\lim_{x \to 0} \frac{\sin x - x \cos x}{x - \sin x}$
30. $\lim_{x \to \pi/2} \left( x - \frac{\pi}{2} \right) \tan x$
31. $\lim_{x \to 0} \frac{\cos(x + \frac{\pi}{2})}{\sin x}$
32. $\lim_{x \to 0} \frac{x^2}{1 - \cos x}$
33. $\lim_{x \to \pi/2} \frac{\cos x}{\sin x}$
34. $\lim_{x \to 0} \frac{\sec x - x}{x^2 - \csc^2 x}$
35. $\lim_{x \to \pi/2} (\sec x - \tan x)$

36. $\lim_{x \to 2} \frac{e^{x^2} - e^4}{x - 2}$
37. $\lim_{x \to 1} \tan \left( \frac{\pi x}{2} \right) \ln x$
38. $\lim_{x \to 1} \frac{x(\ln x - 1) + 1}{(x - 1) \ln x}$
39. $\lim_{x \to 0} \frac{e^{x^2} - 1}{x}$
40. $\lim_{x \to 1} \frac{e^x - e}{\ln x}$
41. $\lim_{x \to 0} \frac{e^{2x} - 1 - x}{x^2}$
42. $\lim_{x \to \infty} \frac{e^{2x} - 1 - x}{x^2}$
43. $\lim_{t \to 0^+} (\sin t)(\ln t)$
44. $\lim_{x \to \infty} e^{-x} (x^3 - x^2 + 9)$
45. $\lim_{x \to 0^+} \frac{a^x - 1}{x}$ $(a > 0)$
46. $\lim_{x \to \infty} x^{1/2^x}$
47. $\lim_{x \to 1} (1 + \ln x)^{1/(x-1)}$
48. $\lim_{x \to 0^+} x^\sin x$
49. $\lim_{x \to 0} \left( \cos x \right)^{3/x^2}$
50. $\lim_{x \to \infty} \left( \frac{x}{x + 1} \right)^x$
51. $\lim_{x \to 0^+} \frac{\sin^{-1} x}{x}$
52. $\lim_{x \to 0^+} \frac{\tan^{-1} x}{\tan \frac{\pi x}{4} - 1}$
53. $\lim_{x \to 1} \frac{\tan^{-1} x - \frac{x}{4}}{\tan \frac{\pi x}{4} - 1}$
54. $\lim_{x \to 0^+} \ln x \tan^{-1} x$

55. Evaluate $x \to \pi/2 \cos n x$, where $m, n \neq 0$ are integers.

56. Eval $\lim_{x \to 1} x^m - 1$ for any numbers $m, n \neq 0$.

57. Prove the following limit formula for $e$:
$e = \lim_{x \to 0} (1 + x)^{1/x}$. Then find a value of $x$ such that $| (1 + x)^{1/x} - e | \leq 0.001$.

58. Can L'Hôpital’s Rule be applied to $\lim_{x \to 0^+} x \sin(1/x)$? Does a graphical or numerical investigation suggest that the limit exists?

59. Let $f(x) = x^{1/x}$ for $x > 0$.
(a) Calculate $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to \infty} f(x)$.
(b) Find the maximum value of \( f(x) \), and determine the intervals on which \( f(x) \) is inc or dec.

60. (a) Use the results of Exercise 59 to prove that \( x^{1/x} = c \) has a unique solution if \( 0 < c \leq 1 \) or \( c = e^{1/e} \), two solutions if \( 1 < c < e^{1/e} \), and no solutions if \( c > e^{1/e} \).

(b) Plot the graph of \( f(x) = x^{1/x} \) and verify that it confirms the conclusions of (a).

61. Determine whether \( f << g \) or \( g << f \) (or neither) for the functions \( f(x) = \log_{10} x \) and \( g(x) = \ln x \).

62. Show that \( x^{1/x} \) has a unique solution if \( 0 < c \leq 1 \) or \( c = e^{1/e} \), two solutions if \( 1 < c < e^{1/e} \), and no solutions if \( c > e^{1/e} \).

63. (a) Use the result of Exercise 59 to prove that \( x^{1/x} \) has a unique solution if \( 0 < c \leq 1 \) or \( c = e^{1/e} \), two solutions if \( 1 < c < e^{1/e} \), and no solutions if \( c > e^{1/e} \).

(b) Plot the graph of \( f(x) = x^{1/x} \) and verify that it confirms the conclusions of (a).

64. Determine whether \( f << g \) or \( g << f \) (or neither) for the functions \( f(x) = \log_{10} x \) and \( g(x) = \ln x \).

65. Show that \( \sqrt{x} << e^{\sqrt{x}} \) or \( e^{\sqrt{x}} << \sqrt{x} \). Hint: Use the substitution \( u = \ln x \) instead of L'Hôpital’s Rule.

66. Show that \( \lim_{x \to \infty} x^{\theta} e^{-x} = 0 \) for all whole numbers \( n > 0 \).

67. Assumptions Matter Let \( f(x) = x (2 + \sin x) \) and \( g(x) = x^2 + 1 \).

(a) Show directly that \( \lim_{x \to \infty} f(x)/g(x) = 0 \).

(b) Show that \( \lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = \infty \), but \( \lim_{x \to \infty} f'(x)/g'(x) \) does not exist.

Do (a) and (b) contradict L'Hôpital’s Rule? Explain.

68. Let \( H(b) = \lim_{x \to \infty} \frac{\ln(1 + b^x)}{x} \) for \( b > 0 \).

(a) Show that \( H(b) = \ln b \) if \( b \geq 1 \)

(b) Determine \( H(b) \) for \( 0 < b \leq 1 \).

69. Let \( G(b) = \lim_{x \to \infty} (1 + b^x)^{1/x} \).

(a) Use the result of Exercise 68 to evaluate \( G(b) \) for all \( b > 0 \).

(b) Verify your result graphically by plotting \( y = (1 + b^x)^{1/x} \) together with the horizontal line \( y = G(b) \) for the values \( b = 0.25, 0.5, 2, 3 \).

70. Show that \( \lim_{t \to \infty} t^k e^{-t^2} = 0 \) for all \( k \). Hint: Compare with \( \lim_{t \to \infty} t^k e^{-t} = 0 \).

In Exercises 71–73, let

\[
\begin{align*}
  f(x) &= \begin{cases} 
    e^{-1/x^2} & \text{for } x \neq 0 \\
    0 & \text{for } x = 0 
  \end{cases}
\end{align*}
\]

These exercises show that \( f(x) \) has an unusual property: All of its derivatives at \( x = 0 \) exist and are equal to zero.

71. Show that \( \lim_{t \to 0} \frac{f(x)}{x^k} = 0 \) for all \( k \). Hint: Let \( t = x^{-1} \) and apply the result of Exercise 70.

72. Show that \( f(t) \) exists and is equal to zero. Also, verify that \( f(t) \) exists and is equal to zero.

73. Show that for \( k \geq 1 \) and \( x \neq 0 \),

\[
f^{(k)}(x) = \frac{P(x)e^{-1/x^2}}{x^r} \quad \text{for some polynomial } P(x) \text{ and some exponent } r \geq 1.
\]

Use the result of Exercise 71 to show that \( f^{(k)}(0) \) exists and is equal to zero for all \( k \geq 1 \).

74. Show that L'Hôpital’s Rule applies to \( \lim_{x \to \infty} \sqrt{x^2 + 1} \) but that it does not help. Then evaluate the limit directly.

75. The Second Derivative Test for critical points fails if \( f''(x) = 0 \). This exercise develops a Higher Derivative Test based on the sign of the first nonzero derivative. Suppose that
\[ f(r) = f_n(r) = \ldots = f^{(n-1)}(r) = 0, \text{ but } f^{(n)}(c) \neq 0 \]

(a) Show, by applying L'Hôpital's Rule \( n \) times, that

\[ \lim_{{x \to c}} \frac{f(x) - f(c)}{x - c} = \frac{1}{n!} f^{(n)}(c) \]

where \( n! = n(n - 1)(n - 2) \ldots (2)(1) \).

(b) Use (a) to show that if \( n \) is even, then \( f(c) \) is a local minimum if \( f^{(n)}(c) > 0 \) and is a local maximum if \( f^{(n)}(c) < 0 \). Hint: If \( n \) is even, then \((x - c)^n > 0 \) for \( x \neq a \), so \( f(x) - f(c) \) must be positive for \( x \) near \( c \) if \( f^{(n)}(c) > 0 \).

(c) Use (a) to show that if \( n \) is odd, then \( f(c) \) is neither a local minimum nor a local maximum.

76. When a spring with natural frequency \( \lambda/2\pi \) is driven with a sinusoidal force \( \sin(\omega t) \) with \( \omega \neq \lambda \), it oscillates according to

\[ y(t) = \frac{1}{\lambda^2 - \omega^2} \left( \lambda \sin(\omega t) - \omega \sin(\lambda t) \right) \]

Let \( y_0(t) = \lim_{{\omega \to \lambda}} y(t) \).

(a) Use L'Hôpital's Rule to determine \( y_0(t) \).

(b) Show that \( y_0(t) \) ceases to be periodic and that its amplitude \( |y_0(t)| \) tends to \( \infty \) as \( t \to \infty \) (the system is said to be in resonance; eventually, the spring is stretched beyond its limits).

(c) Plot \( y(t) \) for \( \lambda = 1 \) and \( \omega = 0.8, 0.9, 0.99 \), and 0.999. Do the graphs confirm your conclusion in (b)?

77. We expended a lot of effort to evaluate

\[ \lim_{{x \to 0}} \frac{\sin x}{x} \quad \text{in Chapter 2.} \]

Show that we could have evaluated it easily using L'Hôpital's Rule. Then explain why this method would involve circular reasoning.

78. By a fact from algebra, if \( f, g \) are polynomials such that \( f(a) = g(a) = 0 \), then there are polynomials \( f_1, g_1 \) such that

\[ f(x) = (x - a)f_1(x), \quad g(x) = (x - a)g_1(x) \]

Use this to verify L'Hôpital's Rule directly for \( \lim_{{x \to a}} \frac{f(x)}{g(x)} \).

79. Patience Required Use L'Hôpital's Rule to evaluate and check your answers numerically:

\[ \lim_{{x \to 0^+}} \left( \frac{\sin x}{x} \right)^{1/x^2} \quad \lim_{{x \to 0}} \left( \frac{1}{\sin^2 x} - \frac{1}{x^2} \right) \]

80. In the following cases, check that \( x = c \) is a critical point and use Exercise 75 to determine whether \( f(c) \) is a local minimum or a local maximum.

(a) \( f(x) = x^5 - 6x^4 + 14x^3 - 16x^2 + 9x + 12 \quad (c = 1) \)

(b) \( f(x) = x^6 - x^3 \quad (c = 0) \)

4.6: Graph Sketching and Asymptotes

1. Determine the sign combinations of \( f' \) and \( f'' \) for each interval \( A-G \) in Figure 16.

2. State the sign change at each transition point \( A-G \) in Figure 17. Ex: \( f'(x) \) goes from + to − at \( A \).
In Exercises 3–6, draw the graph of a function for which \( f \) and \( f' \) take on the given sign combinations.

3. ++, +−, −−
4. ++, +−, −−
5. −+, −−, −+
6. −+, ++, +−

7. Sketch the graph of \( y = x^2 - 5x + 4 \).

8. Sketch the graph of \( y = 12 - 5x - 2x^2 \).

9. Sketch the graph of \( f(x) = x^3 - 3x^2 + 2 \). Include the zeros of \( f(x) \), which are \( x = 1 \) and \( 1 \pm \sqrt{3} \) (approximately \(-0.73, 2.73\)).

10. Show that \( f(x) = x^3 - 3x^2 + 6x \) has a point of inflection but no local extreme values. Sketch.

11. Extend the sketch of the graph of in Example 4 to the interval \([0, 5\pi]\).

12. Sketch the graphs of \( y = x^{2/3} \) and \( y = x^{4/3} \).

In Exercises 13–34, find the transition points, intervals of increase/decrease, concavity, and asymptotic behavior. Then sketch the graph, with this information indicated.

13. \( y = x^3 + 24x^2 \)
14. \( y = x^3 - 3x + 5 \)
15. \( y = x^2 - 4x^3 \)
16. \( y = \frac{1}{3}x^3 + x^2 + 3x \)
17. \( y = 4 - 2x^2 + \frac{1}{6}x^4 \)
18. \( y = 7x^4 - 6x^2 + 1 \)
19. \( y = x^5 + 5x \)
20. \( y = x^5 - 15x^3 \)
21. \( y = x^4 - 3x^3 + 4x \)
22. \( y = x^2(x - 4)^2 \)

23. \( y = x^7 - 14x^6 \)
24. \( y = x^6 - 9x^4 \)
25. \( y = x - 4\sqrt{x} \)
26. \( y = \sqrt{x} + \sqrt{16 - x} \)
27. \( y = x(8 - x)^{1/3} \)
28. \( y = (x^2 - 4x)^{1/3} \)
29. \( y = xe^{-x^2} \)
30. \( y = (2x^2 - 1)e^{-x^2} \)
31. \( y = x - 2\ln x \)
32. \( y = x(4 - x) - 3\ln x \)
33. \( y = x - x^2\ln x \)
34. \( y = x - 2\ln(x^2 + 1) \)

35. Sketch the graph of \( f(x) = 18(x - 3)(x - 1)^{2/3} \) using the formulas

\[
f'(x) = \frac{30(x - \frac{9}{2})}{(x - 1)^{1/3}}, \quad f''(x) = \frac{20(x - \frac{3}{2})}{(x - 1)^{4/3}}
\]

36. Sketch the graph of \( f(x) = \frac{x}{x^2 + 1} \) using the formulas

\[
f'(x) = \frac{1 - x^2}{(1 + x^2)^2}, \quad f''(x) = \frac{2x(x^2 - 3)}{(x^2 + 1)^3}
\]

In Exercises 37–40, sketch the graph of the function, indicating all transition points. If necessary, use a graphing utility or computer algebra system to locate the transition points numerically.

37. \( y = x^2 - 10\ln(x^2 + 1) \)
38. \( y = e^{-x/2}\ln x \)
39. \( y = x^4 - 4x^2 + x + 1 \)
40. \( y = 2\sqrt{x} - \sin x, \quad 0 \leq x \leq 2\pi \)

In Exercises 41–46, sketch the graph over the given interval, with all transition points indicated.

41. \( y = x + \sin x, \quad [0, 2\pi] \)
42. \( y = \sin x + \cos x, \quad [0, 2\pi] \)
43. \( y = 2\sin x - \cos^2 x, \quad [0, 2\pi] \)
44. \( y = \sin x + \frac{1}{2}x, \quad [0, 2\pi] \)
45. \( y = \sin x + \sqrt{3}\cos x, \quad [0, \pi] \)
46. \( y = \sin x - \frac{1}{2} \sin 2x \), \( [0, \pi] \)

47. Are all sign transitions possible? Explain with a sketch why the transitions ++ \( \rightarrow \) +− and −− \( \rightarrow \) +− do not occur if the function is differentiable. (See Exercise 76 for a proof.)

48. Suppose that \( f \) is twice differentiable satisfying (i) \( f(0) = 1 \), (ii) \( f'(x) > 0 \) for all \( x \neq 0 \), and (iii) \( f''(x) < 0 \) for \( x < 0 \) and \( f''(x) > 0 \) for \( x > 0 \). Let \( g(x) = f(x^2) \).

(a) Sketch a possible graph of \( f(x) \).

(b) Prove that \( g(x) \) has no points of inflection and a unique local extreme value at \( x = 0 \). Sketch a possible graph of \( g(x) \).

49. Which of the graphs in Figure 18 cannot be the graph of a polynomial? Explain.

50. Which curve in Figure 19 is the graph of \( f(x) = \frac{2x^4 - 1}{1 + x^4} \)? Explain on the basis of horizontal asymptotes.

51. Match the graphs in Figure 20 with the two functions \( y = \frac{3x}{x^2 - 1} \) and \( y = \frac{3x^2}{x^2 - 1} \). Explain.

52. Match the functions with their graphs in Figure 21.

(a) \( y = \frac{1}{x^2 - 1} \)  (b) \( y = \frac{x^2}{x^2 + 1} \)

(c) \( y = \frac{1}{x^2 + 1} \)  (d) \( y = \frac{x}{x^2 - 1} \)

53. \( y = \frac{1}{3x - 1} \)  54. \( y = \frac{x - 2}{x - 3} \)  55. \( y = \frac{x + 3}{x - 2} \)

56. \( y = x + \frac{1}{x} \)  57. \( y = \frac{1}{x} + \frac{1}{x - 1} \)

58. \( y = \frac{1}{x} - \frac{1}{x - 1} \)  59. \( y = \frac{1}{x(x - 2)} \)
60. \( y = \frac{x}{x^2 - 9} \)  
61. \( y = \frac{1}{x^2 - 6x + 8} \)  
62. \( y = \frac{x^3 + 1}{x} \)  
63. \( y = \frac{1 - \frac{3}{x} + \frac{4}{x^3}}{x^2} \)  
64. \( y = \frac{1}{x^2} + \frac{1}{(x - 2)^2} \)  
65. \( y = \frac{1}{x^2} - \frac{1}{(x - 2)^2} \)  
66. \( y = \frac{4}{x^2 - 9} \)  
67. \( y = \frac{1}{(x^2 + 1)^2} \)  
68. \( y = \frac{x^2}{(x^2 - 1)(x^2 + 1)} \)  
69. \( y = \frac{1}{\sqrt{x^2 + 1}} \)  
70. \( y = \frac{x}{\sqrt{x^2 + 1}} \)

In Exercises 71–75, we explore functions whose graphs approach a nonhorizontal line as \( x \to \infty \). A line \( y = ax + b \) is called a slant asymptote if

\[
\lim_{{x \to \infty}} (f(x) - (ax + b)) = 0 \quad \text{or} \quad \lim_{{x \to -\infty}} (f(x) - (ax + b)) = 0
\]

71. Let \( f(x) = \frac{x^2}{x - 1} \)(Figure 22). Verify the following:

(a) \( f(0) \) is a local max and \( f(2) \) a local min.

(b) \( f \) is con down on \((-\infty, 1)\) and con up on \((1, \infty)\).

(c) \( \lim_{{x \to 1^-}} f(x) = -\infty \) and \( \lim_{{x \to 1^+}} f(x) = \infty \).

(d) \( y = x + 1 \) is a slant asy of \( f(x) \) as \( x \to \pm \infty \).

(e) The slant asymptote lies above the graph of \( f(x) \) for \( x < 1 \) and below the graph for \( x > 1 \).

72. If \( f(x) = \frac{P(x)}{Q(x)} \), where \( P \) and \( Q \) are polynomials of degrees \( m + 1 \) and \( m \), then by long division, we can write \( f(x) = (ax + b) + \frac{P_1(x)}{Q(x)} \) where \( P_1 \) is a polynomial of degree < \( m \). Show that \( y = ax + b \) is the slant asymptote of \( f(x) \). Use this procedure to find the slant asymptotes of the following functions:

\[
(a) \quad y = \frac{x^2 + x}{x^2 + x + 1} \\
(b) \quad \frac{x^2}{x + 1}
\]

73. Sketch the graph of \( f(x) = \frac{x^2}{x + 1} \). Proceed as in the previous exercise to find the slant asymptote.

74. Show that \( y = 3x \) is a slant asymptote for \( f(x) = 3x + x^2 \). Determine whether \( f(x) \) approaches the slant asymptote from above or below and make a sketch of the graph.

75. Sketch the graph of \( f(x) = \frac{1 - x^2}{2 - x} \).

76. Assume that \( f(x) \) and \( f''(x) \) exist for all \( x \) and let \( c \) be a critical point of \( f(x) \). Show that \( f(x) \) cannot make a transition from ++ to −− at \( x = c \). 

\text{Hint: Apply the MVT to } f'(x). \n
77. Assume that \( f''(x) \) exists and \( f''(x) > 0 \) for all \( x \). Show that \( f(x) \) cannot be negative for all \( x \). 

\text{Hint: Show that } f''(b) < 0 \text{ for some } b \text{ and use the result of Exercise 64 in Section 4.4.}

4.7: Applied Optimization
1. Find the dimensions $x$ and $y$ of the rectangle of maximum area that can be formed using 3 meters of wire.

(a) What is the constraint eqn relating $x$ and $y$?

(b) Find a formula for the area in terms of $x$ alone.

(c) What is the interval of optimization? Is it open or closed?

(d) Solve the optimization problem.

2. Wire of length 12 m is divided into two pieces and each piece is bent into a square. How should this be done in order to minimize the sum of the areas of the two squares?

(a) Express the sum of the areas of the squares in terms of the lengths $x$ and $y$ of the two pieces.

(b) What is the constraint eqn relating $x$ and $y$?

(c) What is the interval of optimization? Is it open or closed?

(d) Solve the optimization problem.

3. Wire of length 12 m is divided into two pieces and the pieces are bend into a square and a circle. How should this be done in order to minimize the sum of their areas?

4. Find the positive number $x$ such that the sum of $x$ and its reciprocal is as small as possible. Does this problem require optimization over an open interval or a closed interval?

5. A flexible tube of length 4 m is bent into an $L$-shape. Where should the bend be made to minimize the distance between the two ends?

6. Find the dimensions of the box with square base with:

(a) Volume 12 and the minimal surface area.

(b) Surface area 20 and maximal volume.

7. A rancher will use 600 m of fencing to build a corral in the shape of a semicircle on top of a rectangle (Figure 9). Find the dimensions that maximize the area of the corral.

8. What is the maximum area of a rectangle inscribed in a right triangle with 5 and 8 as in Figure 10. The sides of the rectangle are parallel to the legs of the triangle.

9. Find the dimensions of the rectangle of maximum area that can be inscribed in a circle of radius $r = 4$ (Figure 11).

10. Find the dimensions $x$ and $y$ of the rectangle inscribed in a circle of radius $r$ that maximizes the quantity $xy^2$.

11. Find the point on the line $y = x$ closest to the point $(1, 0)$. Hint: It is equivalent and easier to minimize the square of the distance.

12. Find the point $P$ on the parabola $y = x^2$ closest to the point $(3, 0)$ (Figure 12).

13. Find a good numerical approximation to the coordinates of the point on the graph of $y = \ln x$ closest to the origin (Figure 13).
14. Problem of Tartaglia (1500–1557) Among all positive numbers \(a, b\) whose sum is 8, find those for which the product of the two numbers and their difference is largest.

15. Find the angle \(\theta\) that maximizes the area of the isosceles triangle whose legs have length \(\ell\) (Figure 14).

16. A right circular cone (Figure 15) has volume \(V = \frac{1}{3}\pi r^2 h\) and surface area is \(S = \pi r \sqrt{r^2 + h^2}\). Find the dimensions of the cone with surface area 1 and maximal volume.

17. Find the area of the largest isosceles triangle that can be inscribed in a circle of radius \(r\).

18. Find the radius and height of a cylindrical can of total surface area \(A\) whose volume is as large as possible. Does there exist a cylinder of surface area \(A\) and minimal total volume?

19. A poster of area 6000 cm\(^2\) has blank margins of width 10 cm on the top and bottom and 6 cm on the sides. Find the dimensions that maximize the printed area.

20. According to postal regulations, a carton is classified as “oversized” if the sum of its height and girth (perimeter of its base) exceeds 108 in. Find the dimensions of a carton with square base that is not oversized and has maximum volume.

21. Kepler’s Wine Barrel Problem In his work *Nova stereometria doliorum vinariorum* (New Solid Geometry of a Wine Barrel), published in 1615, astronomer Johannes Kepler stated and solved the following problem: Find the dimensions of the cylinder of largest volume that can be inscribed in a sphere of radius \(R\). *Hint:* Show that an inscribed cylinder has volume \(2\pi x(R^2 - x^2)\), where \(x\) is one-half the height of the cylinder.

22. Find the angle \(\theta\) that maximizes the area of the trapezoid with a base of length 4 and sides of length 2, as in Figure 16.

23. A landscape architect wishes to enclose a rectangular garden of area 1,000 m\(^2\) on one side by a brick wall costing $90/m and on the other three sides by a metal fence costing $30/m. Which dimensions minimize the total cost?

24. The amount of light reaching a point at a distance \(r\) from a light source \(A\) of intensity \(I_A\) is \(I_A/r^2\). Suppose that a second light source \(B\) of intensity \(I_B = 4I_A\) is located 10 m from \(A\). Find the
point on the segment joining \( A \) and \( B \) where the total amount of light is at a minimum.

25. Find the maximum area of a rectangle inscribed in the region bounded by the graph of \( y = \frac{4-x}{2+x} \) and the axes (Figure 17).

![Figure 17](image17)

26. Find the maximum area of a triangle formed by the axes and a tangent line to the graph of \( y = (x + 1)^2 \) with \( x > 0 \).

27. Find the maximum area of a rectangle circumscribed around a rectangle of sides \( L \) and \( H \). *Hint:* Express the area in terms of the angle \( \theta \) (Figure 18).

![Figure 18](image18)

28. A contractor is engaged to build steps up the slope of a hill that has the shape of the graph of \( y = x^2(120-x)/6400 \) for \( 0 \leq x \leq 80 \) with \( x \) in meters (Figure 19). What is the maximum vertical rise of a stair if each stair has a horizontal length of one-third meter?

![Figure 19](image19)

29. Find the equation of the line through \( P = (4, 12) \) such that the triangle bounded by this line and the axes in the first quadrant has minimal area.

30. Let \( P = (a, b) \) lie in the first quadrant. Find the slope of the line through \( P \) such that the triangle bounded by this line and the axes in the first quadrant has minimal area. Then show that \( P \) is the midpoint of the hypotenuse of this triangle.

31. **Archimedes’ Problem** A spherical cap (Figure 20) of radius \( r \) and height \( h \) has volume \( V = \pi h^2(r - \frac{1}{2}h) \) and surface area \( S = 2\pi rh \).

Prove that the hemisphere encloses the largest volume among all spherical caps of fixed surface area \( S \).

32. Find the isosceles triangle of smallest area (Figure 21) that circumscribes a circle of radius 1 (from Thomas Simpson’s *The Doctrine and Application of Fluxions*, a calculus text that appeared in 1750).

![Figure 20](image20)
33. A box of volume $72 \text{ m}^3$ with square bottom and no top is constructed out of two different materials. The cost of the bottom is $40/\text{m}^2$ and the cost of the sides is $30/\text{m}^2$. Find the dimensions of the box that minimize total cost.

34. Find the dimensions of a cylinder of volume $1 \text{ m}^3$ of minimal cost if the top and bottom are made of material that costs twice as much as the material for the side.

35. Your task is to design a rectangular industrial warehouse consisting of three separate spaces of equal size as in Figure 22. The wall materials cost $500 per linear meter and your company allocates $2,400,000 for the project.

(a) Which dimensions maximize the area of the warehouse?

(b) What is the area of each compartment in this case?

36. Suppose, in the previous exercise, that the warehouse consists of $n$ separate spaces of equal size. Find a formula in terms of $n$ for the maximum possible area of the warehouse.

37. According to a model developed by economists E. Heady and J. Pesek, if fertilizer made from $N$ pounds of nitrogen and $\pi$ pounds of phosphate is used on an acre of farmland, then the yield of corn (in bushels per acre) is $Y = 7.5 + 0.6N + 0.7P - 0.001N^2 - 0.002P^2 + 0.001NP$. A farmer intends to spend $30 per acre on fertilizer. If nitrogen costs 25 cents/lb and phosphate costs 20 cents/lb, which combination of $N$ and $L$ produces the highest yield of corn?

38. Experiments show that the quantities $x$ of corn and $y$ of soybean required to produce a hog of weight $Q$ satisfy $Q = 0.5x^{1/2}y^{1/4}$. The unit of $x$, $y$, and $Q$ is the cwt, an agricultural unit equal to 100 lbs. Find the values of $x$ and $y$ that minimize the cost of a hog of weight $Q = 2.5$ cwt if corn costs $3/\text{cwt}$ and soy costs $7/\text{cwt}$.

39. All units in a 100-unit apartment building are rented out when the monthly rent is set at $r = 900/\text{month}$. Suppose that one unit becomes vacant with each $10$ increase in rent and that each occupied unit costs $80/\text{month}$ in maintenance. Which rent $r$ maximizes monthly profit?

40. An 8-billion-bushel corn crop brings a price of $2.40/\text{bu}$. A commodity broker uses the rule of thumb: If the crop is reduced by $x$ percent, then the price increases by $10x$ cents. Which crop size results in maximum revenue and what is the price per bu? *Hint:* Revenue is equal to price times crop size.

41. The monthly output of a Spanish light bulb factory is $P = 2LK^2$ (in millions), where $L$ is the cost of labor and $K$ is the cost of equipment (in millions of euros). The company needs to produce 1.7 million units per month. Which values of $L$ and $K$ would minimize the total cost $L + K$?

42. The rectangular plot in Figure 23 has size $100 \text{ m} \times 200 \text{ m}$. Pipe is to be laid from $A$ to a point $P$ on side $BC$ and from there to $C$. The cost of laying pipe along the side of the plot is $45/\text{m}$ and the cost through the plot is $80/\text{m}$ (since it is underground).
(a) Let $f(x)$ be the total cost, where $x$ is the distance from $P$ to $B$. Determine $f(x)$, but note that $f$ is discontinuous at $x = 0$ (when $x = 0$, the cost of the entire pipe is $45/ft$).

(b) What is the most economical way to lay the pipe? What if the cost along the sides is $65/m$?

43. Brandon is on one side of a river that is 50 m wide and wants to reach a point 200 m downstream on the opposite side as quickly as possible by swimming diagonally across the river and then running the rest of the way. Find the best route if Brandon can swim at 1.5 m/s and run at 4 m/s.

44. **Snell’s Law** When a light beam travels from a point $A$ above a swimming pool to a point $B$ below the water (Figure 24), it chooses the path that takes the least time. Let $v_1$ be the velocity of light in air and $v_2$ the velocity in water (it is known that $v_1 > v_2$). Prove Snell’s Law of Refraction:

$$\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}$$

45. Find the value of $h$ that maximizes the volume of the box if $A = 15$ and $B = 24$. What are the dimensions of this box?

46. **Vascular Branching** A small blood vessel of radius $r$ branches off at an angle $\theta$ from a larger vessel of radius $R$ to supply blood along a path from $A$ to $B$. According to Poiseuille’s Law, the total resistance to blood flow is proportional to

$$T = \left( \frac{a - b \cot \theta}{R^4} + \frac{b \csc \theta}{r^4} \right)$$

where $a$ and $b$ are as in Figure 25. Show that the total resistance is minimized when $\cos \theta = (r/R)^4$.

47. Which values of $A$ and $B$ maximize the volume of the box if $h = 10$ cm and $AB = 900$ cm.

48. Given $n$ numbers $x_1, \ldots, x_n$, find the value of $x$ minimizing the sum of the squares:

$$(x - x_1)^2 + (x - x_2)^2 + \cdots + (x - x_n)^2$$

First solve for $n = 2, 3$ and then try it for arbitrary $n$.

49. A billboard of height $b$ is mounted on the side of a building with its bottom edge at a distance $h$ from the street as in Figure 27. At what distance $x$
should an observer stand from the wall to maximize the angle of observation \( \theta \)?

**50.** Solve Exercise 49 again using geometry rather than calculus. There is a unique circle passing through points \( B \) and \( C \) which is tangent to the street. Let \( R \) be the point of tangency. Note that the two angles labeled \( \psi \) in Figure 27 are equal because they subtend equal arcs on the circle.

(a) Show that the maximum value of \( \theta \) is \( \theta = \psi \).

*Hint:* Show that \( \psi = \theta + \angle PBA \) where \( A \) is the intersection of the circle with \( PC \).

(b) Prove that this agrees with the answer to Exercise 49.

(c) Show that \( \angle QRB = \angle RCQ \) for the maximal angle \( \psi \).

**51. Optimal Delivery Schedule** A gas station sells \( Q \) gallons of gasoline per year, which is delivered \( N \) times per year in equal shipments of \( Q/N \) gallons. The cost of each delivery is \( d \) dollars and the yearly storage costs are \( sQT \), where \( T \) is the length of time (a fraction of a year) between shipments and \( s \) is a constant. Show that costs are minimized for \( N = \sqrt{sQ/d} \). (*Hint:* \( T = 1/N \).) Find the optimal number of deliveries if \( Q = 2 \) million gal, \( d = 8,000 \), and \( s = 30 \) cents/gal-yr. Your answer should be a whole number, so compare costs for the two integer values of \( N \) nearest the optimal value.

**52. Victor Klee’s Endpoint Maximum Problem**

Given 40 meters of straight fence, your goal is to build a rectangular enclosure using 80 additional meters of fence that encompasses the greatest area. Let \( A(x) \) be the area of the enclosure, with \( x \) as in Figure 28.

(a) Find the maximum value of \( A(x) \).

(b) Which interval of \( x \) values is relevant to our problem? Find the maximum value of \( A(x) \) on this interval.

**53.** Let \( (a, b) \) be a fixed point in the first quadrant and let \( S(d) \) be the sum of the distances from \((d, 0)\) to the points \((0, 0), (a, b), \) and \((a, -b)\).

(a) Find the value of \( d \) for which \( S(d) \) is minimal. The answer depends on whether \( b < \sqrt{3}a \) or \( b \geq \sqrt{3}a \). *Hint:* Show that \( d = 0 \) when \( b \geq \sqrt{3}a \).

(b) Let \( a = 1 \). Plot \( S(d) \) for \( b = 0.5, \sqrt{3}, 3 \) and describe the position of the minimum.

**54.** The force \( F \) (in Newtons) required to move a box of mass \( m \) kg in motion by pulling on an attached rope (Figure 29) is

\[
F(\theta) = \frac{fmg}{\cos \theta + \sqrt{m^2 + f^2} \sin \theta}
\]
where $\theta$ is the angle between the rope and the horizontal, $f$ is the coefficient of static friction, and $g = 9.8 \text{ m/s}^2$. Find the angle $\theta$ that minimizes the required force $F$, assuming $f = 0.4$. *Hint:* Find the maximum value of $\cos \theta + f \sin \theta$.

**FIGURE 29**

55. In the setting of Exercise 54, show that for any $f$ the minimal force required is proportional to $1/\sqrt{1 + f^2}$.

56. **Bird Migration** Ornithologists have found that the power (in joules per second) consumed by a certain pigeon flying at velocity $v \text{ m/s}$ is described well by the function $P(v) = 17v^{-1} + 10^{-3}v^3 \text{ J/s}$. Assume that the pigeon can store $5 \times 10^4 \text{ J}$ of usable energy as body fat.

(a) Show that at velocity $v$, a pigeon can fly a total distance of $D(v) = (5 \times 10^4) v / P(v)$ if it uses all of its stored energy.

(b) Find the velocity $v_p$ that *minimizes* $P(v)$.

(c) Migrating birds are smart enough to fly at the velocity that maximizes distance traveled rather than minimizes power consumption. Show that the velocity $v_d$ which maximizes $D(v)$ satisfies $P'(v_d) = P(v_d)/v_d$. Show that $v_d$ is obtained graphically as the velocity coordinate of the point where a line through the origin is tangent to the graph of $P(v)$ (Figure 30).

(d) Find $v_d$ and the maximum distance $D(v_d)$.

57. The problem is to put a “roof” of side $s$ on an attic room of height $h$ and width $b$. Find the smallest length $s$ for which this is possible if $b = 27$ and $h = 8$ (Figure 31).

**FIGURE 30**

58. Redo Exercise 57 for arbitrary $b$ and $h$.

59. Find the maximum length of a pole that can be carried horizontally around a corner joining corridors of widths $a = 24$ and $b = 3$ (Figure 32).

**FIGURE 31**

**FIGURE 32**

59. Find the maximum length of a pole that can be carried horizontally around a corner joining corridors of widths $a = 24$ and $b = 3$ (Figure 32).

60. Redo Exercise 59 for arbitrary widths $a$ and $b$.

61. Find the minimum length $\ell$ of a beam that can clear a fence of height $h$ and touch a wall located $b$ ft behind the fence (Figure 33).
62. Which value of $h$ maximizes the volume of the box if $A = B$?

63. A basketball player stands $d$ feet from the basket. Let $h$ and $\alpha$ be as in Figure 34. Using physics, one can show that if the player releases the ball at an angle $\theta$, then the initial velocity required to make the ball go through the basket satisfies

$$v^2 = \frac{16d}{\cos^2 \theta (\tan \theta - \tan \alpha)}$$

(a) Explain why this formula is meaningful only for $\alpha < \theta < \frac{\pi}{2}$. Why does $v$ approach infinity at the endpoints of this interval?

(b) Take $\alpha = \frac{\pi}{6}$ and plot $v^2$ as a function of $\theta$ for $\frac{\pi}{6} < \theta < \frac{\pi}{2}$. Verify that the minimum occurs at $\theta = \frac{\pi}{3}$.

(c) Set $F(\theta) = \cos^2 \theta (\tan \theta - \tan \alpha)$. Explain why $v$ is minimized for $\theta$ such that $F(\theta)$ is maximized.

(d) Verify that $F'(\theta) = \cos(\alpha - 2\theta) \sec \alpha$ (you will need to use the addition formula for cosine) and show that the maximum value of $F(\theta)$ on $[\alpha, \frac{\pi}{2}]$ occurs at $\theta_0 = \frac{\alpha}{2} + \frac{\pi}{4}$.

(e) For a given $\alpha$, the optimal angle for shooting the basket is $\theta_0$ because it minimizes $v^2$ and therefore minimizes the energy required to make the shot (energy is proportional to $v^2$). Show that the velocity $v_{\text{opt}}$ at the optimal angle $\theta_0$ satisfies

$$v_{\text{opt}}^2 = \frac{32d \cos \alpha}{1 - \sin \alpha} = \frac{32d^2}{-h + \sqrt{d^2 + h^2}}$$

(f) Show with a graph that for fixed $d$ (say, $d = 15$ ft, the distance of a free throw), $v_{\text{opt}}^2$ is an increasing function of $h$. Use this to explain why taller players have an advantage and why it can help to jump while shooting.

64. Three towns $A$, $B$, and $C$ are to be joined by an underground fiber cable as illustrated in Figure 35(A). Assume that $C$ is located directly below the midpoint of $AB$. Find the junction point $P$ that minimizes the total amount of cable used.

(a) First show that $P$ must lie directly above $C$. Hint: Use the result of Example 6 to show that if the junction is placed at point $Q$ in Figure 35(B), then we can reduce the cable length by moving $Q$ horizontally over to the point $P$ lying above $C$.

(b) With $x$ as in Figure below (A), let $f(x)$ be the total length of cable used. Show that $f(x)$ has a unique critical point $c$. Compute $c$ and show that $0 \leq c \leq L$ if and only if $D \leq 2\sqrt{3}L$.

(c) Find the minimum of $f(x)$ on $[0, L]$ in two cases: $D = 2$, $L = 4$ and $D = 8$, $L = 2$. 

\[ \text{FIGURE 33} \]

\[ \text{FIGURE 34} \]

\[ \text{FIGURE 35} \]
65. Tom and Ali drive along a highway represented by the graph of \( f(x) \) in Figure 36. During the trip, Ali views a billboard represented by the segment \( BC \) along the y-axis. Let \( Q \) be the y-intercept of the tangent line to \( y = f(x) \). Show that \( \theta \) is maximized at the value of \( x \) for which the angles \( \angle QPB \) and \( \angle QCP \) are equal. This generalizes Exercise 50 (c) (which corresponds to the case \( f(x) = 0 \)).

**Hints:**

(a) Show that \( d\theta/dx \) is equal to

\[
(b - c) \cdot \frac{(x^2 + (xf'(x))^2) - (b - (f(x) - xf'(x))(c - f(x)))}{(x^2 + (b - f(x))^2)(x^2 + (c - f(x))^2)}
\]

(b) Show that the y-coordinate of \( Q \) is \( f(x) - xf' \).

(c) Show that the condition \( d\theta/dx = 0 \) is equivalent to

\[ PQ^2 = BQ \cdot CQ \]

(d) Conclude that \( \triangle QPB \) and \( \triangle QCP \) are similar triangles.

**Seismic Prospecting** Exercises 66–68 are concerned with determining the thickness \( d \) of a layer of soil that lies on top of a rock formation. Geologists send two sound pulses from point \( A \) to point \( D \) separated by a distance \( s \). The first pulse travels directly from \( A \) to \( D \) along the surface of the earth. The second pulse travels down to the rock formation, then along its surface, and then back up to \( D \) (path \( ABCD \)), as in Figure 37. The pulse travels with velocity \( v_1 \) in the soil and \( v_2 \) in the rock.

66. (a) Show that the time required for the first pulse to travel from \( A \) to \( D \) is \( t_1 = s/v_1 \).

(b) Show that the time required for the second pulse is

\[
t_2 = \frac{2d}{v_1} \sec \theta + \frac{s - 2d \tan \theta}{v_2}
\]

provided that

\[
\tan \theta \leq \frac{s}{2d}
\]

(Note: If this inequality is not satisfied, then point \( B \) does not lie to the left of \( C \).)

(c) Show that \( t_2 \) is minimized when \( \sin \theta = v_1/v_2 \).

67. In this exercise, assume that \( v_2/v_1 \geq \sqrt{1 + 4(d/s)^2} \).

(a) Show that inequality (2) holds if \( \sin \theta = v_1/v_2 \).

(b) Show that the minimal time for the second pulse is

\[
t_2 = \frac{2d}{v_1} (1 - k^2)^{1/2} + \frac{s}{v_2}
\]

where \( k = u_1/u_2 \).

(c) Conclude that \( t_1 = \frac{2d(1 - k^2)^{1/2}}{s} + k \).

68. Continue with the assumption of the previous exercise.

(a) Find the thickness of the soil layer, assuming that \( v_1 = 0.7v_2 \), \( t_2/t_1 = 1.3 \), and \( s = 400 \) m.

(b) The times \( t_1 \) and \( t_2 \) are measured experimentally. The equation in Exercise 67(c) shows that \( t_2/t_1 \) is a linear function of \( 1/s \). What might you conclude if experiments were formed
for several values of \( s \) and the points \( (1/s, t_2/t_1) \) did not lie on a straight line?

![FIG 37]

69. In this exercise we use Figure 38 to prove Heron’s principle of Example 6 without calculus. By definition, \( C \) is the reflection of \( B \) across the line \( MN \) (so that \( BC \) is perpendicular to \( MN \) and \( BN = CN \)). Let \( P \) be the intersection of \( AC \) and \( MN \). Use geometry to justify:

(a) \( \triangle PNB \) and \( \triangle PNC \) are congruent and \( \theta_1 = \theta_2 \).

(b) The paths \( APB \) and \( APC \) have equal length.

(c) Similarly \( AQB \) and \( AQC \) have equal length.

(d) Path \( APC \) is shorter than \( AQC \) for all \( Q \neq P \).

Conclude that the shortest path \( AQB \) occurs for \( Q = P \).

![FIGURE 38]

70. A jewelry designer plans to incorporate a component made of gold in the shape of a frustum of a cone of height 1 cm and fixed lower radius \( r \) (Figure 39). The upper radius \( x \) can take on any value between 0 and \( r \). Note that \( x = 0 \) and \( x = r \) correspond to a cone and cylinder, respectively. As a function of \( x \), the surface area (not including the top and bottom) is \( S(x) = \pi s(r + x) \), where \( s \) is the slant height as indicated in the figure. Which value of \( x \) yields the least expensive design [the minimum value of \( S(x) \) for \( 0 \leq x \leq r \)]?

(a) Show that \( S(x) = \pi(r + x)\sqrt{1 + (r - x)^2} \).

(b) Show that if \( r < \sqrt{2} \), then \( S(x) \) is an increasing function. Conclude that the cone \( (x = 0) \) has minimal area in this case.

(c) Assume that \( r > \sqrt{2} \). Show that \( S(x) \) has two critical points \( x_1 < x_2 \) in \((0, r)\), and that \( S(x_1) \) is a local maximum, and \( S(x_2) \) is a local minimum.

(d) Conclude that the min occurs at \( x = 0 \) or \( x_2 \).

(e) Find the min in the cases \( r = 1.5 \) and \( r = 2 \).

(f) Challenge: Let \( c = (5 + 3\sqrt{3})/4 \approx 1.597 \). Prove that the minimum occurs at \( x = 0 \) (cone) if \( \sqrt{2} < r < c \), but the minimum occurs at \( x = x_2 \) if \( r > c \).

![FIGURE 39]

4.8: Newton’s Method

In this exercise set, all approximations should be carried out using Newton’s Method.

In Exercises 1–6, apply Newton’s Method to \( f(x) \) and initial guess \( x_0 \) to calculate \( x_1, x_2, x_3 \).

1. \( f(x) = x^2 - 6, x_0 = 2 \)  
2. \( f(x) = x^2 - 3x + 1, x_0 = 3 \)

3. \( f(x) = x^3 - 10, x_0 = 2 \)  
4. \( f(x) = x^3 + x + 1, x_0 = -1 \)

5. \( f(x) = \cos x - 4x, x_0 = 1 \)

6. \( f(x) = 1 - x \sin x, x_0 = 7 \)
7. Use Figure 6 to choose an initial guess $x_0$ to the unique real root of $x^3 + 2x + 5 = 0$ and compute the first three Newton iterates.

![Figure 6](image.png)

8. Approximate a solution of $\sin x = \cos 2x$ in the interval $[0, \frac{\pi}{2}]$ to three decimal places. Then find the exact solution and compare with your approximation.

9. Approximate both solutions of $e^x = 5x$ to three decimal places (Figure 7).

![Figure 7](image.png)

10. The first positive solution of $\sin x = 0$ is $x = \pi$. Use Newton’s Method to calculate $\pi$ to four decimal places.

In Exercises 11–14, approximate to three decimal places using Newton’s Method and compare with the value from a calculator.

11. $\sqrt{11}$ 12. $5^{1/3}$ 13. $2^{7/3}$ 14. $3^{-1/4}$

15. Approximate the largest positive root of $f(x) = x^4 - 6x^2 + x + 5$ to within an error of at most $10^{-4}$. Refer to Figure 5.

In Exercises 16–19, approximate the root specified to three decimal places using Newton’s Method. Use a plot to choose an initial guess.

16. Largest positive root of $f(x) = x^3 - 5x + 1$.

17. Negative root of $f(x) = x^5 - 20x + 10$.

18. Positive solution of $\sin \theta = 0.8\theta$.

19. Solution of $\ln(x + 4) = x$.

20. Let $x_1$, $x_2$ be the estimates to a root obtained by applying Newton’s Method with $x_0 = 1$ to the function graphed in Figure 8. Estimate the numerical values of $x_1$ and $x_2$, and draw the tangent lines used to obtain them.

![Figure 8](image.png)

21. Find the smallest positive value of $x$ at which $y = x$ and $y = \tan x$ intersect. Hint: Draw a plot.

22. In 1535, the mathematician Antonio Fior challenged his rival Niccolo Tartaglia to solve this problem: A tree stands 12 braccia high; it is broken into two parts at such a point that the height of the part left standing is the cube root of the length of the part cut away. What is the height of the part left standing? Show that this is equivalent to solving $x^3 + x = 12$ and find the height to three decimal places. Tartaglia, who had discovered the secret of the cubic equation, was able to determine the exact answer:

$$x = \left(\frac{1}{3} \sqrt[3]{2,919 + 54} - \frac{1}{3} \sqrt[3]{2,919 - 54}\right) / \sqrt[3]{6}$$

23. Find (to two decimal places) the coordinates of the point $P$ in Figure 9 where the tangent line to $y = \cos x$ passes through the origin.
Newton’s Method is often used to determine interest rates in financial calculations. In Exercises 24–26, \( r \) denotes a yearly interest rate expressed as a decimal (rather than as a percent).

24. If \( P \) dollars are deposited every month in an account earning interest at the yearly rate \( r \), then the value \( S \) of the account after \( N \) years is

\[
S = P \left( \frac{b^{12N+1} - b}{b - 1} \right) \quad \text{where} \quad b = 1 + \frac{r}{12}
\]

You have decided to deposit \( P = 100 \) dollars per month.

(a) Determine \( S \) after 5 years if \( r = 0.07 \) (that is, 7%).

(b) Show that to save $10,000 after 5 years, you must earn interest at a rate \( r \) determined by the equation \( b^{61} - 101b + 100 = 0 \). Use Newton’s Method to solve for \( b \). Then find \( r \). Note that \( b = 1 \) is a root, but you want the root satisfying \( b > 1 \).

25. If you borrow \( L \) dollars for \( N \) years at a yearly interest rate \( r \), your monthly payment of \( P \) dollars is calculated using the equation

\[
L = P \left( \frac{1 - b^{-12N}}{b - 1} \right) \quad \text{where} \quad b = 1 + \frac{r}{12}
\]

(a) Find \( P \) if \( L = 5,000 \), \( N = 3 \), and \( r = 0.08 \) (8%).

(b) You are offered a loan of \( L = 5,000 \) to be paid back over 3 years with monthly payments of \( P = 200 \). Use Newton’s Method to compute \( b \) and find the implied interest rate \( r \) of this loan. Hint: Show that \((L/P)b^{12N+1} - (1 + L/P)b^{12N} + 1 = 0\).

26. If you deposit \( P \) dollars in a retirement fund every year for \( N \) years with the intention of then withdrawing \( Q \) dollars per year for \( M \) years, you must earn interest at a rate \( r \) satisfying \( P \left( b^N - 1 \right) = Q \left( 1 - b^{-M} \right) \), where \( b = 1 + r \). Assume that $2,000 is deposited each year for 30 years and the goal is to withdraw $10,000 per year for 25 years. Use Newton’s Method to compute \( b \) and then find \( r \). Note that \( b = 1 \) is a root, but you want the root satisfying \( b > 1 \).

27. There is no simple formula for the position at time \( t \) of a planet \( P \) in its orbit (an ellipse) around the sun. Introduce the auxiliary circle and angle \( \theta \) in Figure 10 (note that \( P \) determines \( \theta \) because it is the central angle of point \( B \) on the circle). Let \( a = OA \) and \( e = OS/OA \) (the eccentricity of the orbit).

(a) Show that sector \( BSA \) has area \((a^2/2)(\theta - e \sin \theta)\).

(b) By Kepler’s Second Law, the area of sector \( BSA \) is proportional to the time \( t \) elapsed since the planet passed point \( A \), and because the circle has area \( \pi a^2 \), \( BSA \) has area \((\pi a^2)(t/T)\), where \( T \) is the period of the orbit. Deduce Kepler’s Equation:

\[
\frac{2\pi t}{T} = \theta - e \sin \theta
\]

(c) The eccentricity of Mercury’s orbit is approximately \( e = 0.2 \). Use Newton’s Method to find after a quarter of Mercury’s year has elapsed \((t = T/4)\). Convert \( \theta \) to degrees. Has Mercury covered more than a quarter of its orbit at \( t = T/4 \)?
The roots of \( f(x) = \frac{1}{3}x^3 - 4x + 1 \) to three decimal places are \(-3.583, 0.251, \) and \(3.332\) (Figure 11). Determine the root to which Newton’s Method converges for the initial choices \(x_0 = 1.85, 1.7, \) and \(1.55.\) The answer shows that a small change in \(x_0\) can have a significant effect on the outcome of Newton’s Method.

What happens when you apply Newton’s Method to find a zero of \( f(x) = x^{1/3}? \) Note that \(x = 0\) is the only zero.

Newton’s Method can be used to compute reciprocals without performing division. Let \(c > 0\) and set \(f(x) = x^{-1} - c.\)

(a) Show that \(x - (f(x)/f'(x)) = 2x - cx^2.\)

(b) Calculate the first three iterates of Newton’s Method with \(c = 10.3\) and the two initial guesses \(x_0 = 0.1\) and \(x_0 = 0.5.\)

(c) Explain graphically why \(x_0 = 0.5\) does not yield a sequence converging to \(1/10.3.\)

In Exercise 32 and 33, consider a metal rod of length \(L\) fastened at both ends. If you cut the rod and weld on an additional segment of length \(m,\) leaving the ends fixed, the rod will bow up into a circular arc of radius \(R\) (unknown), as indicated in Figure 12.

Let \(h\) be the max vert displacement of the rod.

(a) Show that \(L = 2R \sin \theta\) and conclude that
\[
\frac{L(1 - \cos \theta)}{2 \sin \theta} = \frac{L}{L + m}
\]

(b) Show that \(L + m = 2R \theta\) and then prove
\[
\frac{\sin \theta}{\theta} = \frac{L}{L + m}
\]

Let \(L = 3\) and \(m = 1.\) Apply Newton’s Method to Eq. (2) to estimate, and use this to estimate \(h.\)

In Exercise 35–37, a flexible chain of length \(L\) is suspended between two poles of equal height separated by a distance \(2M\) (Figure 13). By Newton’s laws, the chain describes a catenary \(y = a \cosh \left(\frac{M}{a}\right),\) where \(a\) is the number such that \(L = 2a \sinh \left(\frac{M}{a}\right).\) The sag \(s\) is the vertical distance from the highest to the lowest point on the chain.

Suppose that \(L = 120\) and \(M = 50.\)

(a) Use Newton’s Method to find a value of \(a\) (to two decimal places) satisfying \(L = 2a \sinh(M/a).\)
36. Assume that \( M \) is fixed.

(a) Calculate \( \frac{ds}{da} \). Note that \( s = a \cosh \left( \frac{M}{a} \right) - a \).

(b) Calculate \( \frac{da}{dl} \) by implicit differentiation using the relation \( L = 2a \sinh \left( \frac{M}{a} \right) \).

(c) Use (a) and (b) and the Chain Rule to show that
\[
\frac{ds}{dl} = \frac{ds}{da} \frac{da}{dl} = \frac{\cosh(M/a) - (M/a) \sinh(M/a)}{2 \sinh(M/a) - (2M/a) \cosh(M/a)}.
\]

37. Suppose that \( L = 160 \) and \( M = 50 \).

(a) Use Newton’s Method to find a value of \( a \) (to two decimal places) satisfying \( L = 2a \sinh(M/a) \).

(b) Use Eq. (3) and the Linear Approximation to estimate the increase in sag \( \Delta s \) for changes in length \( \Delta L = 1 \) and \( \Delta L = 5 \).

(c) Compute \( s(161) - s(160) \) and \( s(165) - s(160) \) directly and compare with your estimates in (b).

FIGURE 13 Chain hanging between two poles.

4.9: Antiderivatives

In Exercises 1–8, find the general antiderivative of \( f(x) \) and check your answer by differentiating.

1. \( f(x) = 18x^2 \)  
2. \( f(x) = x^{3/5} \)
3. \( f(x) = 2x^4 - 24x^2 + 12x^{-1} \)  
4. \( f(x) = 9x + 15x^{-2} \)
5. \( f(x) = 2 \cos x - 9 \sin x \)  
6. \( f(x) = 4x^7 - 3 \cos x \)
7. \( f(x) = 12e^x - 5x^2 \)  
8. \( f(x) = e^x - 4 \sin x \)
9. Match functions (a)–(d) with their antiderivatives (i)–(iv).

(a) \( f(x) = \sin x \)  
(i) \( F(x) = \cos(1 - x) \)
(b) \( f(x) = x \sin(x^2) \)  
(ii) \( F(x) = -\cos x \)
(c) \( f(x) = \sin(1 - x) \)  
(iii) \( F(x) = -\frac{1}{2} \cos(x^2) \)
(d) \( f(x) = x \sin x \)  
(iv) \( F(x) = \sin x - x \cos x \)

In Exercises 10–39, evaluate the indefinite integral.

10. \( \int (9x + 2) \, dx \)  
11. \( \int (4 - 18x) \, dx \)
12. \( \int x^{-3} \, dx \)  
13. \( \int t^{-6/11} \, dt \)
14. \( \int (5t^3 - t^{-3}) \, dt \)  
15. \( \int (18t^5 - 10t^4 - 28t) \, dt \)
16. \( \int 14s^{9/5} \, ds \)  
17. \( \int (z^{-4/5} - z^{2/3} + z^{5/4}) \, dz \)
18. \( \int \frac{3}{2} \, dx \)  
19. \( \int \frac{1}{3\sqrt{x}} \, dx \)  
20. \( \int \frac{dx}{x^{4/3}} \)
21. \( \int \frac{36 \, dt}{t^3} \)  
22. \( \int x(x^2 - 4) \, dx \)
23. \( \int (t^{1/2} + 1)(t + 1) \, dt \)  
24. \( \int \frac{12 - z}{\sqrt{x}} \, dz \)
25. \( \int \frac{x^3 + 3x - 4}{x^2} \, dx \)
26. \( \int \left( \frac{1}{3} \sin x - \frac{1}{4} \cos x \right) \, dx \)
27. \( \int 12 \sec x \tan x \, dx \)  
28. \( \int (\theta + \sec^2 \theta) \, d\theta \)
29. \( \int (\csc t \cot t) \, dt \)  
30. \( \int \sin(7x - 5) \, dx \)

31. \( \int \sec^2(7 - 3\theta) \, d\theta \)  
32. \( \int (\theta - \cos(1 - \theta)) \, d\theta \)

33. \( \int 25 \sec^2(3z + 1) \, dz \)  
34. \( \int \sec(x + 5) \tan(x + 5) \, dx \)

35. \( \int \left( \cos(3\theta) - \frac{1}{2} \sec^2 \left( \frac{\theta}{4} \right) \right) \, d\theta \)

36. \( \int \left( \frac{4}{x} - e^x \right) \, dx \)  
37. \( \int (3e^{5x}) \, dx \)

38. \( \int e^{3t-4} \, dt \)  
39. \( \int (8x^3 - 4e^{5-2x}) \, dx \)

40. In Figure 3, is graph (A) or graph (B) the graph of an antiderivative of \( f(x) \)?

41. In Figure 4, which of graphs (A), (B), and (C) is not the graph of an antiderivative of \( f(x) \)? Explain.

42. Show that \( F(x) = \frac{1}{3}(x + 13)^3 \) is an antiderivative of \( f(x) = (x + 13)^2 \).

In Exercises 43–46, verify by differentiation.

43. \( \int (x + 13)^6 \, dx = \frac{1}{7}(x + 13)^7 + C \)

44. \( \int (x + 13)^{-5} \, dx = -\frac{1}{4}(x + 13)^{-4} + C \)

45. \( \int (4x + 13)^2 \, dx = \frac{1}{12}(4x + 13)^3 + C \)

46. \( \int (ax + b)^n \, dx = \frac{1}{a(n + 1)}(ax + b)^{n+1} + C \) \( \text{for } n \neq -1 \)

In Exercises 47–62, solve the initial value problem.

47. \( \frac{dy}{dx} = x^3, \quad y(0) = 4 \)

48. \( \frac{dy}{dt} = 3 - 2t, \quad y(0) = -5 \)

49. \( \frac{dy}{dt} = 2t + 9t^2, \quad y(1) = 2 \)

50. \( \frac{dy}{dx} = 8x^3 + 3x^2, \quad y(2) = 0 \)

51. \( \frac{dy}{dt} = \sqrt{t}, \quad y(1) = 1 \)

52. \( \frac{dz}{dt} = t^{-3/2}, \quad z(4) = -1 \)

53. \( \frac{dy}{dx} = (3x + 2)^3, \quad y(0) = 1 \)

54. \( \frac{dy}{dt} = (4t + 3)^{-2}, \quad y(1) = 0 \)

55. \( \frac{dy}{dx} = \sin x, \quad y \left( \frac{\pi}{2} \right) = 1 \)

56. \( \frac{dy}{dz} = \sin 2z, \quad y \left( \frac{\pi}{4} \right) = 4 \)
57. \( \frac{dy}{dx} = \cos 5x \), \( y(\pi) = 3 \)

58. \( \frac{dy}{dx} = \sec^2 3x \), \( y \left( \frac{\pi}{4} \right) = 2 \)

59. \( \frac{dy}{dx} = e^x \), \( y(2) = 0 \)

60. \( \frac{dy}{dt} = e^{-t} \), \( y(0) = 0 \)

61. \( \frac{dy}{dt} = 9e^{12-3t} \), \( y(4) = 7 \)

62. \( \frac{dy}{dt} = t + 2e^{t-9} \), \( y(9) = 4 \)

In Exercises 63–69, first find \( f' \) and then find \( f \).

63. \( f \left( x \right) = 12x \), \( f'(0) = 1 \), \( f(0) = 2 \)

64. \( f \left( x \right) = x^3 - 2x \), \( f'(1) = 0 \), \( f(1) = 2 \)

65. \( f \left( x \right) = x^3 - 2x + 1 \), \( f'(0) = 1 \), \( f(0) = 0 \)

66. \( f \left( x \right) = x^3 - 2x + 1 \), \( f'(1) = 0 \), \( f(1) = 4 \)

67. \( f \left( t \right) = t^{-3/2} \), \( f'(4) = 1 \), \( f(4) = 4 \)

68. \( f \left( \theta \right) = \cos \theta \), \( f'(\frac{\pi}{2}) = 1 \), \( f \left( \frac{\pi}{2} \right) = 6 \)

69. \( f \left( t \right) = t - \cos t \), \( f'(0) = 2 \), \( f(0) = -2 \)

70. Show that \( F \left( x \right) = \tan^2 x \) and \( G \left( x \right) = \sec^2 x \) have the same derivative. What can you conclude about the relation between \( F \) and \( G \)? Verify this conclusion directly.

71. A particle located at the origin at \( t = 1 \) s moves along the \( x \)-axis with velocity \( v(t) = (6t^2 - t) \) m/s. State the differential equation with initial condition satisfied by the position \( s(t) \) of the particle, and find \( s(t) \).

72. A particle moves along the \( x \)-axis with velocity \( v(t) = (6t^2 - t) \) m/s. Find the particle’s position \( s(t) \) assuming that \( s(2) = 4 \).

73. A mass oscillates at the end of a spring. Let \( s(t) \) be the displacement of the mass from the equilibrium position at time \( t \). Assuming that the mass is located at the origin at \( t = 0 \) and has velocity \( v(t) = \sin(t/2) \) m/s, state the differential equation with initial condition satisfied by \( s(t) \), and find \( s(t) \).

74. Beginning at \( t = 0 \) with initial velocity \( 4 \) m/s, a particle moves in a straight line with acceleration \( a(t) = 3t^{1/2} \) m/s\(^2\). Find the distance traveled after 25 seconds.

75. A car traveling \( 25 \) m/s begins to decelerate at a constant rate of \( 4 \) m/s\(^2\). After how many seconds does the car come to a stop and how far will the car have traveled before stopping?

76. At time \( t = 1 \) s, a particle is traveling at \( 72 \) m/s and begins to decelerate at the rate \( a(t) = -t^{-1/2} \) until it stops. How far does the particle travel before stopping?

77. A 900-kg rocket is released from a space station. As it burns fuel, the rocket’s mass decreases and its velocity increases. Let \( v(m) \) be the velocity (in meters per second) as a function of mass \( m \). Find the velocity when \( m = 729 \) if \( \frac{dv}{dm} = -50m^{-1/2} \). Assume that \( v(900) = 0 \).

78. As water flows through a tube of radius \( R = 10 \) cm, the velocity \( v \) of an individual water particle depends only on its distance \( r \) from the center of the tube. The particles at the walls of the tube have zero velocity and \( \frac{dv}{dr} = -0.06r \). Determine \( v(r) \).

79. Verify the linearity properties of the indefinite integral stated in Theorem 4.

80. Find constants \( c_1 \) and \( c_2 \) such that \( F(x) = c_1 \sin x + c_2 \cos x \) is an antiderivative of \( f(x) = x \) \( \cos x \).

81. Find constants \( c_1 \) and \( c_2 \) such that \( F(x) = c_1xe^x + c_2e^x \) is an antiderivative of \( f(x) = xe^x \).
82. Suppose that \( F(x) = f(x) \) and \( G(x) = g(x) \). Is it true that \( F(x)G(x) \) is an antiderivative of \( f(x)g(x) \)? Confirm or provide a counterexample.

83. Suppose that \( F(x) = f(x) \).

(a) Show that \( \frac{1}{2} F(2x) \) is an antiderivative of \( f(2x) \).

(b) Find the general antiderivative of \( f(kx) \) for \( k \neq 0 \).

84. Find an antiderivative for \( f(x) = |x| \).

85. Using Theorem 1, prove that \( F(x) = f(x) \) where \( f(x) \) is a polynomial of degree \( n - 1 \), then \( f(x) \) is a polynomial of degree \( n \). Then prove that if \( g(x) \) is any function such that \( g^{(n)}(x) = 0 \), then \( g(x) \) is a polynomial of degree at most \( n \).

86. Show that \( \frac{x^{n+1} - 1}{n+1} \) is an antiderivative of \( y = x^n \) for \( n \neq -1 \). Then use L'Hôpital's Rule to prove that \( \lim_{n \to -1} F(x) = \ln x \).

In Exercises 13–18, use the Linear Approximation.

13. The position of an object in linear motion at time \( t \) is \( s(t) = 0.4t^2 + (t + 1)^{-1} \). Estimate the distance traveled over the time interval \([4, 4.2]\).

14. A bond that pays $10,000 in 6 years is offered for sale at a price \( P \). The percentage yield \( Y \) of the bond is verified that if \( P = 7,500 \), then \( Y = 4.91\% \). Estimate the drop in yield if the price rises to $7,700.

15. When a bus pass from Albuquerque to Los Alamos is priced at \( p \) dollars, a bus company takes in a monthly revenue of \( R(p) = 1.5p - 0.01p^2 \) (in thousands of dollars).

(a) Estimate \( \Delta R \) if the price rises from $50 to $53.

(b) If \( p = 80 \), how will revenue be affected by a small increase in price? Explain using the Linear Approximation.

16. A store sells 80 MP4 players per week when the players are priced at \( P = 75 \). Estimate the number \( N \) sold if \( P \) is raised to $80, assuming that \( dN/dP = -4 \). Est. \( N \) if the price is lowered to $69.

17. The circumference of a sphere is measured at \( C = 100 \) cm. Estimate the maximum percentage error in \( V \) if the error in \( C \) is at most 3 cm.

18. Show that \( \sqrt{a^2 + b} \approx a + \frac{b}{2a} \) if \( b \) is small. Use this to estimate \( \sqrt{26} \) and find the error using a calculator.

4Review

In Exercises 1–6, estimate using the Linear Approximation or linearization, and use a calculator to estimate the error.

1. \( 8.1^{1/3} - 2 \)
2. \( \frac{1}{\sqrt{4.1}} - \frac{1}{2} \)
3. \( 625^{1/4} - 624^{1/4} \)
4. \( \sqrt{101} \)
5. \( \frac{1}{1.02} \)
6. \( \frac{5}{33} \)

In Exercises 7–12, find the linearization at the point indicated.

7. \( y = \sqrt{x}, \; a = 25 \)
8. \( v(t) = 32t - 4t^2, \; a = 2 \)

9. \( A(r) = \frac{4}{3} \pi r^3, \; a = 3 \)
10. \( V(h) = 4h(2 - h)(4 - 2h), \; a = 1 \)
11. \( P(x) = e^{-x^2/2}, \; a = 1 \)
12. \( f(x) = \ln(x + e), \; a = e \)
19. Use the Intermediate Value Theorem to prove that \( \sin x - \cos x = 3x \) has a solution, and use Rolle’s Theorem to show that this solution is unique.

20. Show that \( f(x) = 2x^3 + 2x + \sin x + 1 \) has precisely one real root.

21. Verify the MVT for \( f(x) = \ln x \) on \([1, 4]\).

22. Suppose that \( f(1) = 5 \) and \( f'(x) \geq 2 \) for \( x \geq 1 \). Use the MVT to show that \( f(x) \geq 2x + 4 \) for all \( x \geq 0 \).

23. A function \( f(x) \) has derivative \( f'(x) = \frac{1}{x^4 + 1} \). Where on the interval \([1, 4]\) does \( f(x) \) take on its maximum value?

24. In Exercises 25–30, find the critical points and determine whether they are minima, maxima, or neither.

25. \( f(x) = x^3 - 4x^2 + 4x \)

26. \( s(t) = t^4 - 8t^2 \)

27. \( f(x) = x^2(x + 2)^3 \)

28. \( f(x) = x^{23}(1 - x) \)

29. \( g(\theta) = \sin^2 \theta + \theta \)

30. \( h(\theta) = 2 \cos 2\theta + \cos 4\theta \)

31. \( f(x) = x(10 - x), [-1, 3] \)

32. \( f(x) = 6x^4 - 4x^6, [-2, 2] \)

33. \( g(\theta) = \sin^2 \theta - \cos \theta, [0, 2\pi] \)

34. \( R(t) = \frac{t}{t^2 + t + 1}, [0, 3] \)

35. \( f(x) = x^{23/2} - 2x^{1/3}, [-1, 3] \)

36. \( f(x) = 4x - \tan^2 x, \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \)

37. \( f(x) = x - 12 \ln x, [5, 40] \)

38. \( f(x) = e^x - 20x - 1, [0, 5] \)

39. Find the critical points and extreme values of \( f(x) = |x - 1| + |2x - 6| \) in \([0, 8]\).

40. Match the description of \( f(x) \) with the graph of its derivative \( f'(x) \) in Figure 1.

(a) \( f(x) \) is increasing and concave up.

(b) \( f(x) \) is decreasing and concave up.

(c) \( f(x) \) is increasing and concave down.

41. \( y = x^3 - 4x^2 + 4x \)

42. \( y = x - 2\cos x \)

43. \( y = \frac{x^2}{x^2 + 4} \)

44. \( y = \frac{x}{(x^2 - 4)^{1/3}} \)

45. \( f(x) = (x^2 - x)e^{-x} \)

46. \( f(x) = x(\ln x)^2 \)

47. \( y = 12x - 3x^2 \)

48. \( y = 8x^2 - x^4 \)

49. \( y = x^3 - 2x^2 + 3 \)

50. \( y = 4x - x^{3/2} \)

51. \( y = \frac{x}{x^2 + 1} \)

52. \( y = \frac{x}{(x^2 - 4)^{2/3}} \)

53. \( y = \frac{1}{|x + 2| + 1} \)

54. \( y = \sqrt{2 - x^3} \)

55. \( y = \sqrt{3} \sin x - \cos x \) on \([0, 2\pi]\)

56. \( y = 2x - \tan x \) on \([0, 2\pi]\)
57. Draw a curve \( y = f(x) \) for which \( f' \) and \( f'' \) have signs as indicated in Figure 2.

\[
\begin{array}{c|cccccc}
- & + & - & - & + & + & - \\
-2 & 0 & 1 & 3 & 5 & x
\end{array}
\]

FIGURE 2

58. Find the dimensions of a cylindrical can with a bottom but no top of volume 4 m\(^3\) that uses the least amount of metal.

59. A rectangular box of height \( h \) with square base of side \( b \) has volume \( V = 4 \) m\(^3\). Two of the side faces are made of material costing $40/m\(^2\). The remaining sides cost $20/m\(^2\). Which values of \( b \) and \( h \) minimize the cost of the box?

60. The corn yield on a certain farm is

\[ Y = -0.118x^2 + 8.5x + 12.9 \quad \text{(bushels per acre)} \]

where \( x \) is the number of corn plants per acre (in thousands). Assume that corn seed costs $1.25 (per thousand seeds) and that corn can be sold for $1.50/bushel. Let \( P(x) \) be the profit (revenue minus the cost of seeds) at planting level \( x \).

(a) Compute \( P(x_0) \) for the value \( x_0 \) that maximizes yield \( Y \).

(b) Find the maximum value of \( P(x) \). Does maximum yield lead to maximum profit?

61. Let \( N(t) \) be the size of a tumor (in units of \( 10^6 \) cells) at time \( t \) (in days). According to the Gompertz Model, \( dN/dt = N(a - b \ln N) \) where \( a, b \) are positive constants. Show that the maximum value of \( N \) is \( e^{ab} \) and that the tumor increases most rapidly when \( N = e^{ab}/e \).

62. A truck gets 10 miles per gallon of diesel fuel traveling along an interstate highway at 50 mph. This mileage decreases by 0.15 mpg for each mile per hour increase above 50 mph. 

(a) If the truck driver is paid $30/hour and diesel fuel costs \( P = $3/gal \), which speed \( v \) between 50 and 70 mph will minimize the cost of a trip along the highway? Notice that the actual cost depends on the length of the trip, but the optimal speed does not.

(b) Plot cost as a function of \( v \) (choose the length arbitrarily) and verify your answer to part (a).

(c) Do you expect the optimal speed \( v \) to increase or decrease if fuel costs go down to \( P = $2/gal \)? Plot the graphs of cost as a function of \( v \) for \( P = 2 \) and \( P = 3 \) on the same axis and verify your conclusion.

63. Find the maximum volume of a right-circular cone placed upside-down in a right-circular cone of radius \( R = 3 \) and height \( H = 4 \) as in Figure 3. A cone of radius \( r \) and height \( h \) has volume \( \frac{1}{3} \pi r^2 h \).

64. Redo Exercise 63 for arbitrary \( R \) and \( H \).

65. Show that the maximum area of a parallelogram \( ADEF \) that is inscribed in a triangle \( ABC \), as in Figure 4, is equal to one-half the area of \( \triangle ABC \).
66. A box of volume 8 m$^3$ with a square top and bottom is constructed out of two types of metal. The metal for the top and bottom costs $50/m^2$ and the metal for the sides costs $30/m^2$. Find the dimensions of the box that minimize total cost.

67. Let $f(x)$ be a function whose graph does not pass through the $x$-axis and let $Q = (a, 0)$. Let $P = (x_0, f(x_0))$ be the point on the graph closest to $Q$ (Figure 5). Prove that $\overrightarrow{PQ}$ is perpendicular to the tangent line to the graph of $x_0$. Hint: Find the minimum value of the square of the distance from $(x, f(x))$ to $(a, 0)$.

68. Take a circular piece of paper of radius $R$, remove a sector of angle $\theta$ (Figure 6), and fold the remaining piece into a cone-shaped cup. Which angle $\theta$ produces the cup of largest volume?

69. Use Newton’s Method to estimate $\sqrt[3]{25}$ to four decimal places.

70. Use Newton’s Method to find a root of $f(x) = x^2 - x - 1$ to four decimal places.

In Exercises 71–84, calculate the indefinite integral.

71. $\int (4x^3 - 2x^2) \, dx$
72. $\int x^{9/4} \, dx$
73. $\int \sin(\theta - 8) \, d\theta$
74. $\int \cos(5 - 7\theta) \, d\theta$
75. $\int (4t^{-3} - 12t^{-4}) \, dt$
76. $\int (9t^{-2/3} + 4t^{7/3}) \, dt$
77. $\int \sec^2 x \, dx$
78. $\int \tan 3\theta \sec 3\theta \, d\theta$
79. $\int (y + 2)^4 \, dy$
80. $\int \frac{3x^3 - 9}{x^2} \, dx$
81. $\int (e^x - x) \, dx$
82. $\int e^{-4x} \, dx$
83. $\int 4x^{-1} \, dx$
84. $\int \sin(4x - 9) \, dx$

In Exercises 85–90, solve the differential equation with the given initial condition.

85. $\frac{dy}{dx} = 4x^3$, $y(1) = 4$
86. $\frac{dy}{dt} = 3t^2 + \cos t$, $y(0) = 12$
87. $\frac{dy}{dx} = x^{-1/2}$, $y(1) = 1$
88. $\frac{dy}{dx} = \sec^2 x$, $y(\frac{\pi}{4}) = 2$
89. $\frac{dy}{dx} = e^{-x}$, $y(0) = 3$
90. \[ \frac{dy}{dx} = e^{4x}, \quad y(1) = 1 \]

91. Find \( f(t) \) if \( f(t) = 1 - 2t, f(0) = 2, \) and \( f'(0) = -1. \)

92. At time \( t = 0, \) a driver begins decelerating at a constant rate of \(-10 \text{ m/s}^2\) and comes to a halt after traveling 500 m. Find the velocity at \( t = 0. \)

93. Find the local extrema of \( f(x) = \frac{e^{2x} + 1}{e^x+1}. \)

94. Find the points of inflection of \( f(x) = \ln(x^2 + 1), \) and at each point, determine whether the concavity changes from up to down or from down to up.

In Exercises 95–98, find the local extrema and points of inflection, and sketch the graph. Use L’Hôpital’s Rule to determine the limits as \( x \to 0^+ \) or \( x \to \pm \infty \) if necessary.

95. \( y = x \ln x \quad (x > 0) \quad 96. \quad y = e^{x-x^2} \)

97. \( y = x \ln(x^2) \quad (x > 0) \quad 98. \quad y = \tan^{-1}\left(\frac{x^2}{4}\right) \)

99. Explain why L’Hôpital’s Rule gives no information about \( \lim_{x \to \infty} \frac{2x - \sin x}{3x + \cos 2x} \). Evaluate the limit by another method.

100. Let \( f(x) \) be a differentiable function with inverse \( g(x) \) such that \( f(0) = 0 \) and \( f'(0) \neq 0. \) Prove that

\[ \lim_{x \to 0} \frac{f(x)}{g(x)} = f'(0)^2 \]

In Exercises 101–112, verify that L’Hôpital’s Rule applies and evaluate the limit.

101. \( \lim_{x \to 3} \frac{4x - 12}{x^2 - 5x + 6} \quad 102. \quad \lim_{x \to \infty} \frac{x^3 + 2x^2 - x - 2}{2x^4 + 2x^3 - 4x - 8} \)

103. \( \lim_{x \to 0^+} \frac{x^{1/2} \ln x}{\ln(e^x + 1)} \quad 104. \quad \lim_{t \to \infty} \frac{\ln(e^t + 1)}{t} \)

105. \( \lim_{\theta \to 0} \frac{2 \sin \theta - 2 \sin 2\theta}{\sin \theta - \theta \cos \theta} \)

106. \( \lim_{x \to 0} \frac{\sqrt{4 + x} - 2 \sqrt{x}}{x^2} \quad 107. \quad \lim_{t \to \infty} \frac{\ln(t + 2)}{\log_2 t} \)

108. \( \lim_{x \to 0} \frac{\left(\frac{e^x}{e^x - 1} - \frac{1}{x}\right)}{\ln(x^2)} \quad 109. \quad \lim_{y \to 0^+} \frac{y - \sin^{-1} y - y^3}{y^3} \quad 110. \quad \lim_{x \to 1} \frac{\sqrt{1 - x^2}}{\cos^{-1} x} \)

111. \( \lim_{x \to 0} \frac{\sinh(x^2)}{\cosh x - 1} \quad 112. \quad \lim_{x \to 0} \frac{\tanh x - \sinh x}{\sin x - x} \)

113. Let \( f(x) = e^{-Ax^2/2}, \) where \( A > 0. \) Given any \( n \) numbers \( a_1, a_2, \ldots, a_n, \) set

\[ \Phi(x) = f(x - a_1) f(x - a_2) \cdots f(x - a_n) \]

(a) Assume \( n = 2 \) and prove that \( \Phi(x) \) attains its maximum value at the average \( x = \frac{1}{2}(a_1 + a_2). \)

Hint: Calculate \( \Phi'(x) \) using logarithmic differentiation.

(b) Show that for any \( n, \) \( \Phi(x) \) attains its maximum value at \( x = \frac{1}{n}(a_1 + a_2 + \cdots + a_n). \) This fact is related to the role of \( f(x) \) (whose graph is a bell-shaped curve) in statistics.