6.1 Area Between Two Curves

Preliminary Questions

1. What is the area interpretation of \( \int_a^b (f(x) - g(x)) \, dx \) if \( f(x) \geq g(x) \)?

SOLUTION Because \( f(x) \geq g(x) \), \( \int_a^b (f(x) - g(x)) \, dx \) represents the area of the region bounded between the graphs of \( y = f(x) \) and \( y = g(x) \), bounded on the left by the vertical line \( x = a \) and on the right by the vertical line \( x = b \).

2. Is \( \int_a^b (f(x) - g(x)) \, dx \) still equal to the area between the graphs of \( f \) and \( g \) if \( f(x) \geq 0 \) but \( g(x) \leq 0 \)?

SOLUTION Yes. Since \( f(x) \geq 0 \) and \( g(x) \leq 0 \), it follows that \( f(x) - g(x) \geq 0 \).

3. Suppose that \( f(x) \geq g(x) \) on \([0, 3]\) and \( g(x) \geq f(x) \) on \([3, 5]\). Express the area between the graphs over \([0, 5]\) as a sum of integrals.

SOLUTION Remember that to calculate an area between two curves, one must subtract the equation for the lower curve from the equation for the upper curve. Over the interval \([0, 3]\), \( y = f(x) \) is the upper curve. On the other hand, over the interval \([3, 5]\), \( y = g(x) \) is the upper curve. The area between the graphs over the interval \([0, 5]\) is therefore given by

\[
\int_0^3 (f(x) - g(x)) \, dx + \int_3^5 (g(x) - f(x)) \, dx.
\]

4. Suppose that the graph of \( x = f(y) \) lies to the left of the \( y \)-axis. Is \( \int_a^b f(y) \, dy \) positive or negative?

SOLUTION If the graph of \( x = f(y) \) lies to the left of the \( y \)-axis, then for each value of \( y \), the corresponding value of \( x \) is less than zero. Hence, the value of \( \int_a^b f(y) \, dy \) is negative.

Exercises

1. Find the area of the region between \( y = 3x^2 + 12 \) and \( y = 4x + 4 \) over \([-3, 3] \) (Figure 8).

SOLUTION As the graph of \( y = 3x^2 + 12 \) lies above the graph of \( y = 4x + 4 \) over the interval \([-3, 3]\), the area between the graphs is

\[
\int_{-3}^3 (3x^2 + 12) - (4x + 4) \, dx = \int_{-3}^3 (3x^2 - 4x + 8) \, dx = \left( x^3 - 2x^2 + 8x \right)_{-3}^3 = 102.
\]

2. Compute the area of the region in Figure 9(A), which lies between \( y = 2 - x^2 \) and \( y = -2 \) over \([-2, 2]\).
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CHAPTER 6

In Exercises 5–7, find the area between \( y \)

Find the area enclosed by the graphs of \( y \)

Compute the area of the region \( y \)

Thus, the graphs of \( y \)

As the graph of \( y \)

Over the interval \( \pi \)

SOLUTION

Over the interval \( \pi \)

SOLUTION

Over the interval \( \pi \)

In Exercises 5–7, find the area between \( y = \sin x \) and \( y = \cos x \) over the interval. Sketch the curves if necessary.

5. \([0, \frac{\pi}{4}]\)

SOLUTION

Over the interval \([0, \frac{\pi}{4}]\), the graph of \( y = \sin x \) lies below that of \( y = \cos x \). Hence, the area between the two curves is

\[
\int_0^{\pi/4} (\cos x - \sin x) \, dx = (\sin x + \cos x)_{\pi/4}^0 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - (0 + 1) = \sqrt{2} - 1.
\]

6. \([\frac{\pi}{4}, \frac{\pi}{2}]\)

SOLUTION

Over the interval \([\frac{\pi}{4}, \frac{\pi}{2}]\), the graph of \( y = \cos x \) lies below that of \( y = \sin x \). Hence, the area between the two curves is

\[
\int_{\pi/4}^{\pi/2} (\sin x - \cos x) \, dx = (\cos x - \sin x)_{\pi/2}^{\pi/4} = (0 - 1) - \left( \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right) = \sqrt{2} - 1.
\]

7. \([0, \pi]\)

SOLUTION

Over the interval \([0, \frac{\pi}{4}]\), the graph of \( y = \sin x \) lies below that of \( y = \cos x \), while over the interval \([\frac{\pi}{4}, \pi]\), the orientation of the graphs is reversed. The area between the graphs over \([0, \pi]\) is then

\[
\int_0^{\pi/4} (\cos x - \sin x) \, dx + \int_{\pi/4}^{\pi} (\sin x - \cos x) \, dx
\]

\[
= (\sin x + \cos x)_{\pi/4}^{\pi/4} + (\cos x - \sin x)_{0}^{\pi/4} - (\cos x - \sin x)_{\pi/4}^{\pi}
\]

\[
= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - (0 + 1) + (1 - 0) - \left( \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right) = 2\sqrt{2}.
\]
In Exercises 8–10, let \( f(x) = 20 + x - x^2 \) and \( g(x) = x^2 - 5x \).

8. Find the area between the graphs of \( f \) and \( g \) over \([1, 3]\).

**SOLUTION** Setting \( f(x) = g(x) \) gives \( 20 + x - x^2 = x^2 - 5x \), which simplifies to
\[
0 = 2x^2 - 6x - 20 = 2(x - 5)(x + 2).
\]
Thus, the curves intersect at \( x = -2 \) and \( x = 5 \). On the interval \([1, 3]\), the upper curve is \( y = 20 + x - x^2 \). The area between the graphs over the interval \([1, 3]\) is then
\[
\int_1^3 \left( (20 + x - x^2) - (x^2 - 5x) \right) \, dx = \int_1^3 \left( 20 + 6x - 2x^2 \right) \, dx = \left. \left( 20x + 3x^2 - \frac{2}{3}x^3 \right) \right|_1^3 = \frac{140}{3}.
\]

9. Find the area of the region enclosed by the two graphs.

**SOLUTION** Setting \( f(x) = g(x) \) gives \( 20 + x - x^2 = x^2 - 5x \), which simplifies to
\[
0 = 2x^2 - 6x - 20 = 2(x - 5)(x + 2).
\]
Thus, the curves intersect at \( x = -2 \) and \( x = 5 \). With \( y = 20 + x - x^2 \) being the upper curve, the area between the two curves is
\[
\int_{-2}^5 \left( (20 + x - x^2) - (x^2 - 5x) \right) \, dx = \int_{-2}^5 \left( 20 + 6x - 2x^2 \right) \, dx = \left. \left( 20x + 3x^2 - \frac{2}{3}x^3 \right) \right|_{-2}^5 = \frac{343}{3}.
\]

10. Compute the area of the region between the two graphs over \([4, 8]\) as a sum of two integrals.

**SOLUTION** Setting \( f(x) = g(x) \) gives \( 20 + x - x^2 = x^2 - 5x \), which simplifies to
\[
0 = 2x^2 - 6x - 20 = 2(x - 5)(x + 2).
\]
Thus, the curves intersect at \( x = -2 \) and \( x = 5 \). Over the interval \([4, 5]\), \( y = 20 + x - x^2 \) is the upper curve but over the interval \([5, 8]\), \( y = x^2 - 5x \) is the upper curve. The area between the two curves over the interval \([4, 8]\) is then
\[
\int_4^5 \left( (20 + x - x^2) - (x^2 - 5x) \right) \, dx + \int_5^8 \left( (x^2 - 5x) - (20 + x - x^2) \right) \, dx
\]
\[
= \int_4^5 \left( -2x^2 + 6x + 20 \right) \, dx + \int_5^8 \left( 2x^2 - 6x - 20 \right) \, dx
\]
\[
= \left( \frac{-2}{3}x^3 + 3x^2 + 20x \right) \bigg|_4^5 + \left( \frac{2}{3}x^3 - 3x^2 - 20x \right) \bigg|_5^8 = \frac{19}{3} + 81 = 262.\]

11. Find the area between \( y = e^x \) and \( y = e^{2x} \) over \([0, 1]\).

**SOLUTION** As the graph of \( y = e^{2x} \) lies above the graph of \( y = e^x \) over the interval \([0, 1]\), the area between the graphs is
\[
\int_0^1 \left( e^{2x} - e^x \right) \, dx = \left( \frac{1}{2}e^{2x} - e^x \right) \bigg|_0^1 = \frac{1}{2}e^2 - e - \left( \frac{1}{2} - 1 \right) = \frac{1}{2}e^2 - e + \frac{1}{2}.
\]

12. Find the area of the region bounded by \( y = e^x \) and \( y = 12 - e^x \) and the \( y \)-axis.

**SOLUTION** The two graphs intersect when \( e^x = 12 - e^x \), or when \( x = \ln 6 \). As the graph of \( y = 12 - e^x \) lies above the graph of \( y = e^x \) over the interval \([0, \ln 6]\), the area between the graphs is
\[
\int_0^{\ln 6} \left( 12 - e^x - e^x \right) \, dx = \left( 12x - 2e^x \right) \bigg|_0^{\ln 6} = 12\ln 6 - 12 - (0 - 2) = 12\ln 6 - 10.
\]

13. Sketch the region bounded by \( y = \frac{1}{\sqrt{1 - x^2}} \) and \( y = -\frac{1}{\sqrt{1 - x^2}} \) for \(-\frac{1}{2} \leq x \leq \frac{1}{2}\) and find its area.

**SOLUTION** A sketch of the region bounded by \( y = \frac{1}{\sqrt{1 - x^2}} \) and \( y = -\frac{1}{\sqrt{1 - x^2}} \) for \(-\frac{1}{2} \leq x \leq \frac{1}{2}\) is shown below.

As the graph of \( y = \frac{1}{\sqrt{1 - x^2}} \) lies above the graph of \( y = -\frac{1}{\sqrt{1 - x^2}} \), the area between the graphs is
\[
\int_{-1/2}^{1/2} \left( \frac{1}{\sqrt{1 - x^2}} - \left( -\frac{1}{\sqrt{1 - x^2}} \right) \right) \, dx = 2\sin^{-1}x \bigg|_{-1/2}^{1/2} = 2\left[ \frac{\pi}{6} - \left( -\frac{\pi}{6} \right) \right] = \frac{2\pi}{3}.
\]
14. Sketch the region bounded by \( y = \sec^2 x \) and \( y = 2 \) and find its area.

**Solution** A sketch of the region bounded by \( y = \sec^2 x \) and \( y = 2 \) is shown below. Note the region extends from \( x = -\frac{\pi}{4} \) on the left to \( x = \frac{\pi}{4} \) on the right. As the graph of \( y = 2 \) lies above the graph of \( y = \sec^2 x \), the area between the graphs is

\[
\int_{-\pi/4}^{\pi/4} (2 - \sec^2 x) \, dx = \left[ 2x - \tan x \right]_{-\pi/4}^{\pi/4} = \left( \frac{\pi}{2} - 1 \right) - \left( -\frac{\pi}{2} + 1 \right) = \pi - 2.
\]

In Exercises 15–18, find the area of the shaded region in the figure.

15.

**Solution** As the graph of \( y = x^3 - 2x^2 + 10 \) lies above the graph of \( y = 3x^2 + 4x - 10 \), the area of the shaded region is

\[
\int_{-2}^{2} (x^3 - 2x^2 + 10) - (3x^2 + 4x - 10) \, dx = \int_{-2}^{2} (x^3 - 5x^2 - 4x + 20) \, dx = \left( \frac{1}{4}x^4 - \frac{5}{3}x^3 - 2x^2 + 20x \right) \bigg|_{-2}^{2} = \frac{160}{3}.
\]

16.

**Solution** Setting \( x = \sin 2x \) yields \( \sin x (2 \cos x - 1) = 0 \), so the points of intersection are \( x = 0, x = \frac{\pi}{3} \) and \( x = \pi \). Over the interval \([0, \frac{\pi}{3}]\), \( y = \sin 2x \) is the upper curve but over the interval \([\frac{\pi}{3}, \pi]\), \( y = \sin x \) is the upper curve. The area of the shaded region is then

\[
\int_{0}^{\pi/3} (\sin x - \sin 2x) \, dx + \int_{\pi/3}^{\pi} (\sin x - \sin 2x) \, dx = \left( -\frac{1}{2} \cos 2x + \cos x \right) \bigg|_{0}^{\pi/3} + \left( -\cos x + \frac{1}{2} \cos 2x \right) \bigg|_{\pi/3}^{\pi} = \frac{1}{4} + \frac{9}{4} = \frac{5}{2}.
\]
17. \[ y = \frac{1}{2}x \]

**FIGURE 12**

**SOLUTION** Setting \( \frac{1}{2}x = x \sqrt{1 - x^2} \) yields \( x = 0 \) or \( \frac{1}{2} = \sqrt{1 - x^2} \), so that \( x = \pm \frac{\sqrt{3}}{2} \). Over the interval \([-\frac{\sqrt{3}}{2}, 0]\), \( y = \frac{1}{2}x \) is the upper curve but over the interval \([0, \frac{\sqrt{3}}{2}]\), \( y = x \sqrt{1 - x^2} \) is the upper curve. The area of the shaded region is then

\[
\int_{-\frac{\sqrt{3}}{2}}^{0} \left( \frac{1}{2}x - x \sqrt{1 - x^2} \right) dx + \int_{0}^{\frac{\sqrt{3}}{2}} \left( x \sqrt{1 - x^2} - \frac{1}{2}x \right) dx
\]

\[
= \left( \frac{1}{4}x^2 + \frac{1}{3}(1 - x^2)^{3/2} \right)_{-\frac{\sqrt{3}}{2}}^{0} + \left( -\frac{1}{3}(1 - x^2)^{3/2} - \frac{1}{4}x^2 \right)_{0}^{\frac{\sqrt{3}}{2}} = \frac{5}{48} + \frac{5}{48} = \frac{5}{24}.
\]

18. \[ y = \cos x \]

**FIGURE 13**

**SOLUTION** The line on the top-left has equation \( y = \frac{3\sqrt{3}}{2}x \), and the line on the bottom-right has equation \( y = \frac{3}{\pi x} \). Thus, the area to the left of \( x = \frac{\pi}{6} \) is

\[
\int_{0}^{\pi/6} \left( \frac{3\sqrt{3}}{2}x - \frac{3}{2\pi}x \right) dx = \left( \frac{3\sqrt{3}}{2\pi}x^2 - \frac{3}{4\pi}x^2 \right)_{0}^{\pi/6} = \frac{3\sqrt{3} \pi^2}{2\pi} - \frac{3 \pi^2}{4\pi} = \frac{(2\sqrt{3} - 1)\pi}{48}.
\]

The area to the right of \( x = \frac{\pi}{6} \) is

\[
\int_{\pi/6}^{\pi/3} \left( \cos x - \frac{3}{2\pi}x \right) dx = \left( \sin x - \frac{3}{4\pi}x^2 \right)_{\pi/6}^{\pi/3} = \frac{8\sqrt{3} - 8 - \pi}{16}.
\]

The entire area is then

\[
\frac{(2\sqrt{3} - 1)\pi}{48} + \frac{8\sqrt{3} - 8 - \pi}{16} = 12\sqrt{3} - 12 + (\sqrt{3} - 2)\pi.
\]

19. Find the area of the region enclosed by the curves \( y = x^3 - 6x \) and \( y = 8 - 3x^2 \).

**SOLUTION** Setting \( x^3 - 6x = 8 - 3x^2 \) yields \((x + 1)(x + 4)(x - 2) = 0\), so the two curves intersect at \( x = -4 \), \( x = -1 \) and \( x = 2 \). Over the interval \([-4, -1]\), \( y = x^3 - 6x \) is the upper curve, while \( y = 8 - 3x^2 \) is the upper curve over the interval \([-1, 2]\). The area of the region enclosed by the two curves is then

\[
\int_{-4}^{-1} \left( (x^3 - 6x) - (8 - 3x^2) \right) dx + \int_{-1}^{2} \left( (8 - 3x^2) - (x^3 - 6x) \right) dx
\]

\[
= \left( \frac{1}{4}x^4 - 3x^2 - 8x + x^3 \right)_{-4}^{-1} + \left( 8x - x^3 - \frac{1}{4}x^4 + 3x^2 \right)_{-1}^{2} = \frac{81}{4} + \frac{81}{4} = \frac{81}{2}.
\]

20. Find the area of the region enclosed by the semicubical parabola \( y^2 = x^3 \) and the line \( x = 1 \).

**SOLUTION** Since \( y^2 = x^3 \), it follows that \( x \geq 0 \) since \( y^2 \geq 0 \). Therefore, \( y = \pm x^{3/2} \), and the area of the region enclosed by the semicubical parabola and \( x = 1 \) is

\[
\int_{0}^{1} \left( x^{3/2} - (-x^{3/2}) \right) dx = \int_{0}^{1} 2x^{3/2} dx = \left[ \frac{4}{5}x^{5/2} \right]_{0}^{1} = \frac{4}{5}.
\]
In Exercises 21–22, find the area between the graphs of \( x = \sin y \) and \( x = 1 - \cos y \) over the given interval (Figure 14).

![Figure 14](image)

21. \( 0 \leq y \leq \frac{\pi}{2} \)

**SOLUTION** As shown in the figure, the graph on the right is \( x = \sin y \) and the graph on the left is \( x = 1 - \cos y \). Therefore, the area between the two curves is given by

\[
\int_0^{\pi/2} (\sin y - (1 - \cos y)) \, dy = (-\cos y - y + \sin y) \bigg|_0^{\pi/2} = \left( -\frac{\pi}{2} + 1 \right) - (-1) = 2 - \frac{\pi}{2}.
\]

22. \( -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \)

**SOLUTION** The shaded region in the figure shows the area between the graphs from \( y = 0 \) to \( y = \frac{\pi}{2} \). It is bounded on the right by \( x = \sin y \) and on the left by \( x = 1 - \cos y \). Therefore, the area between the graphs from \( y = 0 \) to \( y = \frac{\pi}{2} \) is

\[
\int_0^{\pi/2} (\sin y - (1 - \cos y)) \, dy = (-\cos y - y + \sin y) \bigg|_0^{\pi/2} = \left( -\frac{\pi}{2} + 1 \right) - (-1) = 2 - \frac{\pi}{2}.
\]

The graphs cross at \( y = 0 \). Since \( x = 1 - \cos y \) lies to the right of \( x = \sin y \) on the interval \([-\frac{\pi}{2}, 0]\) along the \( y \)-axis, the area between the graphs from \( y = -\frac{\pi}{2} \) to \( y = 0 \) is

\[
\int_{-\pi/2}^{0} ((1 - \cos y) - \sin y) \, dy = (y - \sin y + \cos y) \bigg|_{-\pi/2}^{0} = 1 - \left( -\frac{\pi}{2} + 1 \right) = \frac{\pi}{2}.
\]

The total area between the graphs from \( y = -\frac{\pi}{2} \) to \( y = \frac{\pi}{2} \) is the sum

\[
\int_0^{\pi/2} (\sin y - (1 - \cos y)) \, dy + \int_{-\pi/2}^{0} ((1 - \cos y) - \sin y) \, dy = 2 - \frac{\pi}{2} + \frac{\pi}{2} = 2.
\]

23. Find the area of the region lying to the right of \( x = y^2 + 4y - 22 \) and the left of \( x = 3y + 8 \).

**SOLUTION** Setting \( y^2 + 4y - 22 = 3y + 8 \) yields

\[
0 = y^2 + y - 30 = (y + 6)(y - 5),
\]

so the two curves intersect at \( y = -6 \) and \( y = 5 \). The area in question is then given by

\[
\int_{-6}^{5} \left( 3y + 8 - (y^2 + 4y - 22) \right) \, dy = \int_{-6}^{5} \left( -y^2 - y + 30 \right) \, dy = \left( -\frac{y^3}{3} - \frac{y^2}{2} + 30y \right) \bigg|_{-6}^{5} = \frac{1331}{6}.
\]

24. Find the area of the region lying to the right of \( x = y^2 - 5 \) and the left of \( x = 3 - y^2 \).

**SOLUTION** Setting \( y^2 + 5 = 3 - y^2 \) yields \( 2y^2 = 8 \) or \( y = \pm 2 \). The area of the region enclosed by the two graphs is then

\[
\int_{-2}^{2} \left( 3 - y^2 \right) \, dy - \int_{-2}^{2} \left( y^2 + 5 \right) \, dy = \int_{-2}^{2} \left( 8 - 2y^2 \right) \, dy = \left( 8y - \frac{2}{3}y^3 \right) \bigg|_{-2}^{2} = \frac{64}{3}.
\]

25. Calculate the area enclosed by \( x = 9 - y^2 \) and \( x = 5 \) in two ways: as an integral along the \( y \)-axis and as an integral along the \( x \)-axis.
SOLUTION Along the y-axis, we have points of intersection at \( y = \pm 2 \). Therefore, the area enclosed by the two curves is

\[
\int_{-2}^{2} (9 - y^2 - 5) \, dy = \int_{-2}^{2} (4 - y^2) \, dy = \left( 4y - \frac{1}{3}y^3 \right) \bigg|_{-2}^{2} = \frac{32}{3}.
\]

Along the x-axis, we have integration limits of \( x = 5 \) and \( x = 9 \). Therefore, the area enclosed by the two curves is

\[
\int_{5}^{9} \frac{2\sqrt{9 - x}}{x} \, dx = -\frac{4}{3} (9 - x)^{3/2} \bigg|_{5}^{9} = 0 - \left( -\frac{32}{3} \right) = \frac{32}{3}.
\]

26. Figure 15 shows the graphs of \( x = y^3 - 26y + 10 \) and \( x = 40 - 6y^2 - y^3 \). Match the equations with the curve and compute the area of the shaded region.

![Figure 15](image_url)

SOLUTION Substituting \( y = 0 \) into the equations for both curves indicates that the graph of \( x = y^3 - 26y + 10 \) passes through the point \((10, 0)\) while the graph of \( x = 40 - 6y^2 - y^3 \) passes through the point \((40, 0)\). Therefore, over the y-interval \([-1, 3]\), the graph of \( x = 40 - 6y^2 - y^3 \) lies to the right of the graph of \( x = y^3 - 26y + 10 \). The orientation of the two graphs is reversed over the y-interval \([-5, -1]\). Hence, the area of the shaded region is

\[
\begin{align*}
\int_{-5}^{-1} (y^3 - 26y + 10) - (40 - 6y^2 - y^3) \, dy &+ \int_{-1}^{3} ((40 - 6y^2 - y^3) - (y^3 - 26y + 10)) \, dy \\
&= \int_{-5}^{-1} (2y^3 + 6y^2 - 26y - 30) \, dy + \int_{-1}^{3} (-2y^3 - 6y^2 + 26y + 30) \, dy \\
&= \left( \frac{1}{2}y^4 + 2y^3 - 13y^2 - 30y \right) \bigg|_{-5}^{-1} + \left( -\frac{1}{2}y^4 - 2y^3 + 13y^2 + 30y \right) \bigg|_{-1}^{3} = 256.
\end{align*}
\]

In Exercises 27–28, find the area of the region using the method (integration along either the x- or y-axis) that requires you to evaluate just one integral.

27. Region between \( y^2 = x + 5 \) and \( y^2 = 3 - x \)

SOLUTION From the figure below, we see that integration along the x-axis would require two integrals, but integration along the y-axis requires only one integral. Setting \( y^2 - 5 = 3 - y^2 \) yields points of intersection at \( y = \pm 2 \). Thus, the area is given by

\[
\int_{-2}^{2} (3 - y^2) - (y^2 + 5) \, dy = \int_{-2}^{2} (8 - 2y^2) \, dy = \left( 8y - \frac{2}{3}y^3 \right) \bigg|_{-2}^{2} = \frac{64}{3}.
\]

![Figure 27](image_url)

28. Region between \( y = x \) and \( x + y = 8 \) over \([2, 3]\)

SOLUTION From the figure below, we see that integration along the y-axis would require three integrals, but integration along the x-axis requires only one integral. The area of the region is then

\[
\int_{2}^{3} (8 - x) - x \, dx = (8x - x^2) \bigg|_{2}^{3} = (24 - 9) - (16 - 4) = 3.
\]

As a check, the area of a trapezoid is given by

\[
\frac{h}{2}(b_1 + b_2) = \frac{1}{2}(4 + 2) = 3.
\]
In Exercises 29–45, sketch the region enclosed by the curves and compute its area as an integral along the x- or y-axis.

29. \(y = 4 - x^2, \quad y = x^2 - 4\)

**Solution** Setting \(4 - x^2 = x^2 - 4\) yields \(2x^2 = 8\) or \(x^2 = 4\). Thus, the curves \(y = 4 - x^2\) and \(y = x^2 - 4\) intersect at \(x = \pm 2\). From the figure below, we see that \(y = 4 - x^2\) lies above \(y = x^2 - 4\) over the interval \([-2, 2]\); hence, the area of the region enclosed by the curves is

\[
\int_{-2}^{2} \left( (4 - x^2) - (x^2 - 4) \right) \, dx = \int_{-2}^{2} (8 - 2x^2) \, dx = \left[ \frac{8x}{3} - \frac{2}{3}x^3 \right]_{-2}^{2} = \frac{64}{3}
\]

30. \(y = x^2 - 6, \quad y = 6 - x^3, \quad y\)-axis

**Solution** Setting \(x^2 - 6 = 6 - x^3\) yields

\[0 = x^3 + x^2 - 12 = (x - 2)(x^2 + 3x + 6),\]

so the curves \(y = x^2 - 6\) and \(y = 6 - x^3\) intersect at \(x = 2\). Using the graph shown below, we see that \(y = 6 - x^3\) lies above \(y = x^2 - 6\) over the interval \([0, 2]\); hence, the area of the region enclosed by these curves and the y-axis is

\[
\int_{0}^{2} \left( (6 - x^3) - (x^2 - 6) \right) \, dx = \int_{0}^{2} (-x^3 + x^2 + 12) \, dx = \left[ \frac{-1}{4}x^4 - \frac{1}{3}x^3 + 12x \right]_{0}^{2} = \frac{52}{3}
\]

31. \(x = \sin y, \quad x = \frac{2}{\pi} y\)

**Solution** Here, integration along the y-axis will require less work than integration along the x-axis. The curves intersect when \(\frac{2}{\pi} y = \sin y\) or when \(y = 0, \pm \frac{\pi}{2}\). From the graph below, we see that both curves are symmetric with respect to the origin. It follows that the portion of the region enclosed by the curves in the first quadrant is identical to the region enclosed in the third quadrant. We can therefore determine the total area enclosed by the two curves by doubling the area enclosed in the first quadrant. In the first quadrant, \(x = \sin y\) lies to the right of \(x = \frac{2}{\pi} y\), so the total area enclosed by the two curves is

\[
2 \int_{0}^{\pi/2} \left( \sin y - \frac{2}{\pi} y \right) \, dy = 2 \left( -\cos y - \frac{1}{\pi} y^2 \right)_{0}^{\pi/2} = 2 \left[ (0 - \frac{\pi}{4}) - (-1 - 0) \right] = 2 - \frac{\pi}{2}
\]
32. \( x + y = 4, \ x - y = 0, \ y + 3x = 4 \)

**SOLUTION** From the graph below, we see that the top of the region enclosed by the three lines is always bounded by \( x + y = 4 \). On the other hand, the bottom of the region is bounded by \( y + 3x = 4 \) for \( 0 \leq x \leq 1 \) and by \( x - y = 0 \) for \( 1 \leq x \leq 2 \). The total area of the region is then

\[
\int_0^1 (4 - x - (4 - 3x)) \, dx + \int_1^2 (4 - x) \, dx = \int_0^1 2x \, dx + \int_1^2 \sqrt{2 \pi} \, dx
\]

\[
= x^2 \bigg|_0^1 (4x - x^2) \bigg|_1^2 = 1 + (8 - 4) - (4 - 1) = 2.
\]

![Diagram of the region enclosed by the three lines](image)

33. \( y = 3x^{-3}, \ y = 4 - x, \ y = \frac{x}{3} \)

**SOLUTION** The curves \( y = 3x^{-3} \) and \( y = 4 - x \) intersect at \( x = 1 \), the curves \( y = 3x^{-3} \) and \( y = x/3 \) intersect at \( x = \sqrt{3} \) and the curves \( y = 4 - x \) and \( y = x/3 \) intersect at \( x = 3 \). From the graph below, we see that the top of the region enclosed by the three curves is always bounded by \( y = 4 - x \). The bottom of the region is bounded by \( y = 3x^{-3} \) for \( 1 \leq x \leq \sqrt{3} \) and by \( y = x/3 \) for \( \sqrt{3} \leq x \leq 3 \). The total area of the region is then

\[
\int_1^{\sqrt{3}} \left(4 - x - 3x^{-3}\right) \, dx + \int_1^3 \left(4 - x - \frac{1}{3x}\right) \, dx = \left(4x - \frac{1}{2}x^2 + \frac{3}{2x^2}\right) \bigg|_1^{\sqrt{3}} + \left(4x - \frac{2}{3x^2}\right) \bigg|_1^3 = 2.
\]

![Diagram of the region enclosed by the three curves](image)

34. \( y = 2 - \sqrt{x}, \ y = \sqrt{x}, \ x = 0 \)

**SOLUTION** Setting \( 2 - \sqrt{x} = \sqrt{x} \) yields \( \sqrt{x} = 1 \) or \( x = 1 \). Using the graph shown below, we see that \( y = 2 - \sqrt{x} \) lies above \( y = \sqrt{x} \) over the interval \([0, 1]\). The area of the region enclosed by these two curves and the \( y \)-axis is then

\[
\int_0^1 \left(2 - \sqrt{x} - \sqrt{x}\right) \, dx = \int_0^1 (2 - 2\sqrt{x}) \, dx = \left(2x - \frac{4}{3}x^{3/2}\right) \bigg|_0^1 = 2.
\]

![Diagram of the region enclosed by the two curves](image)

35. \( y = x\sqrt{x} - 2, \ y = -x\sqrt{x} - 2, \ x = 4 \)

**SOLUTION** Note that \( y = x\sqrt{x} - 2 \) and \( y = -x\sqrt{x} - 2 \) are the upper and lower branches, respectively, of the curve \( y^2 = x^2(x - 2) \). The area enclosed by this curve and the vertical line \( x = 4 \) is

\[
\int_2^4 (x\sqrt{x} - 2 + x\sqrt{x} - 2) \, dx = \int_2^4 2x\sqrt{x} - 2 \, dx.
\]

Substitute \( u = x - 2 \). Then \( du = dx, x = u + 2 \) and

\[
\int_2^4 2x\sqrt{x} - 2 \, dx = \int_0^2 (u + 2)\sqrt{u} \, du = \int_0^2 \left(2u^{3/2} + 4u^{1/2}\right) \, du = \left(\frac{4}{5}u^{5/2} + \frac{8}{3}u^{3/2}\right) \bigg |_0^2 = \frac{128\sqrt{2}}{15}.
\]
36. \( y = |x|, \quad y = x^2 - 6 \)

**SOLUTION**  From the graph below, we see that the region enclosed by the curves \( y = |x| \) and \( y = x^2 - 6 \) is symmetric with respect to the y-axis. We can therefore determine the total area of the region by doubling the area of the portion of the region to the right of the y-axis. For \( x > 0 \), setting \( x = x^2 - 6 \) yields
\[
0 = x^2 - x - 6 = (x - 3)(x + 2),
\]
so the curves intersect at \( x = 3 \). Moreover, on the interval \([0, 3]\), \( y = |x| = x \) lies above \( y = x^2 - 6 \). Therefore, the area of the region enclosed by the two curves is
\[
2 \int_0^3 (x - (x^2 - 6)) \, dx = 2 \left( \frac{1}{2} x^2 - \frac{1}{3} x^3 + 6x \right)_0^3 = 2 \left( \frac{9}{2} - 9 + 18 \right) = 27.
\]

37. \( x = |y|, \quad x = 6 - y^2 \)

**SOLUTION**  From the graph below, we see that integration along the y-axis will require less work than integration along the x-axis. Moreover, the region is symmetric with respect to the x-axis, so the total area can be determined by doubling the area of the upper portion of the region. For \( y > 0 \), setting \( y = 6 - y^2 \) yields
\[
0 = y^2 + y - 6 = (y - 2)(y + 3),
\]
so the curves intersect at \( y = 2 \). Because \( x = 6 - y^2 \) lies to the right of \( x = |y| = y \) in the first quadrant, we find the total area of the region is
\[
2 \int_0^2 (6 - y^2 - y) \, dy = 2 \left( 6y - \frac{1}{3} y^3 - \frac{1}{2} y^2 \right)_0^2 = 2 \left( 12 - \frac{8}{3} - 2 \right) = \frac{44}{3}.
\]

38. \( x = |y|, \quad x = 1 - |y| \)

**SOLUTION**  From the graph below, we see that the region enclosed by the curves \( x = |y| \) and \( x = 1 - |y| \) is symmetric with respect to the x-axis. We can therefore determine the total area by doubling the area in the first quadrant. For \( y > 0 \), setting \( y = 1 - y \) yields \( y = \frac{1}{2} \) as the point of intersection. Moreover, \( x = 1 - |y| = 1 - y \) lies to the right of \( x = |y| = y \), so the total area of the region is
\[
2 \int_0^{1/2} ((1 - y) - y) \, dy = 2(y - y^2)^{1/2}_0 = 2 \left( \frac{1}{2} - \frac{1}{4} \right) = \frac{1}{2}.
\]
39. \( x = 12 - y, \ x = y, \ x = 2y \)

**SOLUTION** From the graph below, we see that the bottom of the region enclosed by the three curves is always bounded by \( y = \frac{x}{2} \). On the other hand, the top of the region is bounded by \( y = x \) for \( 0 \leq x \leq 6 \) and by \( y = 12 - x \) for \( 6 \leq x \leq 8 \).

The area of the region is then

\[
\int_0^6 \left( x - \frac{x}{2} \right) \, dx + \int_6^8 \left( (12 - x) - \frac{x}{2} \right) \, dx = \frac{1}{4} \left| \frac{x^2}{2} \right|^6_0 + \left( 12x - \frac{3}{4} x^2 \right)^8_6 = 9 + (96 - 48) - (72 - 27) = 12.
\]

40. \( x = y^3 - 18y, \ y + 2x = 0 \)

**SOLUTION** Setting \( y^3 - 18y = -\frac{y}{2} \) yields

\[
0 = y^3 - \frac{35}{2} y = y \left( y^2 - \frac{35}{2} \right),
\]

so the points of intersection occur at \( y = 0 \) and \( y = \pm \frac{\sqrt{70}}{2} \). From the graph below, we see that both curves are symmetric with respect to the origin. It follows that the portion of the region enclosed by the curves in the second quadrant is identical to the region enclosed in the fourth quadrant. We can therefore determine the total area enclosed by the two curves by doubling the area enclosed in the second quadrant. In the second quadrant, \( y + 2x = 0 \) lies to the right of \( x = y^3 - 18y \), so the total area enclosed by the two curves is

\[
2 \int_0^{\frac{\sqrt{70}}{2}} \left( \frac{y}{2} - (y^3 - 18y) \right) \, dy = 2 \left( \frac{35}{4} y^2 - \frac{1}{4} y^4 \right)^{\sqrt{70}/2}_0 = 2 \left( \frac{1225}{8} - \frac{1225}{16} \right) = \frac{1225}{8}.
\]

41. \( x = 2y, \ x + 1 = (y - 1)^2 \)

**SOLUTION** Setting \( 2y = (y - 1)^2 - 1 \) yields

\[
0 = y^2 - 4y = y(y - 4),
\]

so the two curves intersect at \( y = 0 \) and at \( y = 4 \). From the graph below, we see that \( x = 2y \) lies to the right of \( x + 1 = (y - 1)^2 \) over the interval \([0, 4]\) along the \( y \)-axis. Thus, the area of the region enclosed by the two curves is

\[
\int_0^4 \left( 2y - ((y - 1)^2 - 1) \right) \, dy = \int_0^4 \left( 4y - y^2 \right) \, dy = \left( 2y^2 - \frac{1}{3} y^3 \right)^4_0 = \frac{32}{3}.
\]

42. \( x + y = 1, \ x^{1/2} + y^{1/2} = 1 \)
**SOLUTION**  From the graph below, we see that the two curves intersect at \( x = 0 \) and at \( x = 1 \) and that \( x + y = 1 \) lies above \( x^{1/2} + y^{1/2} = 1 \). The area of the region enclosed by the two curves is then

\[
\int_0^1 \left( (1 - x) - (1 - \sqrt{x})^2 \right) \, dx = \int_0^1 (-2x + 2\sqrt{x}) \, dx = \left( -x^2 + \frac{4}{3}x^{3/2} \right)_0^1 = \frac{1}{3}.
\]

43. \( y = 6, \ y = x^{-2} + x^2 \) (in the region \( x > 0 \))

**SOLUTION** Setting \( 6 = x^{-2} + x^2 \) yields

\[
0 = x^4 - 6x^2 + 1,
\]

which is a quadratic equation in the variable \( x^2 \). By the quadratic formula,

\[
x^2 = \frac{6 \pm \sqrt{36 - 4}}{2} = 3 \pm 2\sqrt{2}.
\]

Now,

\[
3 + 2\sqrt{2} = (\sqrt{2} + 1)^2 \quad \text{and} \quad 3 - 2\sqrt{2} = (\sqrt{2} - 1)^2,
\]

so the two curves intersect at \( x = \sqrt{2} - 1 \) and \( x = \sqrt{2} + 1 \). Note there are also two points of intersection with \( x < 0 \), but as the problem specifies the region is for \( x > 0 \), we neglect these other two values. From the graph below, we see that \( y = 6 \) lies above \( y = x^{-2} + x^2 \), so the area of the region enclosed by the two curves for \( x > 0 \) is

\[
\int_{\sqrt{2} - 1}^{\sqrt{2} + 1} \left( 6 - (x^{-2} + x^2) \right) \, dx = \left( 6x + x^{-1} - \frac{1}{3}x^3 \right)\bigg|_{\sqrt{2} - 1}^{\sqrt{2} + 1} = \frac{16}{3}.
\]

44. \( y = \cos x, \ y = \cos(2x), \ x = 0, \ x = \frac{2\pi}{3} \)

**SOLUTION** From the graph below, we see that \( y = \cos x \) lies above \( y = \cos 2x \) over the interval \([0, \frac{2\pi}{3}]\). The area of the region enclosed by the two curves is therefore

\[
\int_0^{2\pi/3} \left( \cos x - \cos 2x \right) \, dx = \left( \sin x - \frac{1}{2} \sin 2x \right)_0^{2\pi/3} = \frac{3\sqrt{3}}{4}.
\]

45. \( y = \sin x, \ y = \csc^2 x, \ x = \frac{\pi}{4}, \ x = \frac{3\pi}{4} \)
SOLUTION  Over the interval \([\frac{\pi}{4}, \frac{3\pi}{4}]\), \(y = \csc^2 x\) lies above \(y = \sin x\). The area of the region enclosed by the two curves is then

\[
\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (\csc^2 x - \sin x) \, dx = \left(-\cot x + \cos x\right)\bigg|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = \left(1 - \frac{\sqrt{2}}{2}\right) - \left(-1 + \frac{\sqrt{2}}{2}\right) = 2 - \sqrt{2}.
\]

46. CAS  Plot \(y = \frac{x}{\sqrt{x^2 + 1}}\) and \(y = (x - 1)^2\) on the same set of axes. Use a computer algebra system to find the points of intersection numerically and compute the area between the curves.

SOLUTION  Using a computer algebra system, we find that the curves

\[y = \frac{x}{\sqrt{x^2 + 1}}\quad\text{and}\quad y = (x - 1)^2\]

intersect at \(x = 0.3943285581\) and at \(x = 1.942944418\). From the graph below, we see that \(y = \frac{x}{\sqrt{x^2 + 1}}\) lies above \(y = (x - 1)^2\), so the area of the region enclosed by the two curves is

\[
\int_{0.3943285581}^{1.942944418} \left(\frac{x}{\sqrt{x^2 + 1}} - (x - 1)^2\right) \, dx = 0.7567130951
\]

The value of the definite integral was also obtained using a computer algebra system.

47. Sketch a region whose area is represented by

\[
\int_{-\sqrt{2}/2}^{\sqrt{2}/2} \left(\sqrt{1 - x^2} - |x|\right) \, dx
\]

and evaluate using geometry.

SOLUTION  Matching the integrand \(\sqrt{1 - x^2} - |x|\) with the \(y_{\text{TOP}} - y_{\text{BOT}}\) template for calculating area, we see that the region in question is bounded along the top by the curve \(y = \sqrt{1 - x^2}\) (the upper half of the unit circle) and is bounded along the bottom by the curve \(y = |x|\). Hence, the region is \(\frac{1}{4}\) of the unit circle (see the figure below). The area of the region must then be

\[
\frac{1}{4} \pi(1)^2 = \frac{\pi}{4}.
\]
Two athletes run in the same direction along a straight track with velocities \( v_1(t) \) and \( v_2(t) \) (in ft/s). Assume that

\[
\int_0^5 (v_1(t) - v_2(t)) \, dt = 2, \quad \int_0^{20} (v_1(t) - v_2(t)) \, dt = 5 \\
\int_{30}^{35} (v_1(t) - v_2(t)) \, dt = -2
\]

(a) Give a verbal interpretation of the integral \( \int_{t_1}^{t_2} (v_1(t) - v_2(t)) \, dt \).

(b) Is enough information given to determine the distance between the two runners at time \( t = 5 \) s?

(c) If the runners begin at the same time and place, how far ahead is runner 1 at time \( t = 20 \) s?

(d) Suppose that runner 1 is 8 ft ahead at \( t = 30 \) s. How far is she ahead or behind at \( t = 35 \) s?

SOLUTION

(a) The integral \( \int_{t_1}^{t_2} (v_1(t) - v_2(t)) \, dt \) represents the difference in the distance traveled by the two runners from time \( t_1 \) to time \( t_2 \).

(b) Because we do not know where the runners started, we cannot determine the distance between the two runners at time \( t = 5 \) seconds. However, knowing that

\[
\int_0^5 (v_1(t) - v_2(t)) \, dt = 2 \text{ ft},
\]

we may conclude the following. If runner 1 started ahead of runner 2, runner 1 is 2 feet further ahead at time \( t = 5 \) seconds, whereas if runner 1 started more than 2 feet behind runner 2, runner 1 is 2 feet closer at time \( t = 5 \) seconds. Finally, if runner 1 started fewer than 2 feet behind runner 2, then runner 1 has passed runner 2 at time \( t = 5 \) seconds.

(c) Because

\[
\int_0^{20} (v_1(t) - v_2(t)) \, dt = 5 \text{ ft},
\]

runner 1 is 5 feet ahead of runner 2 at time \( t = 20 \) seconds.

(d) Because

\[
\int_{30}^{35} (v_1(t) - v_2(t)) \, dt = -2 \text{ ft},
\]

runner 2 has gotten 2 feet closer to runner 1 during the time interval from \( t = 30 \) seconds to \( t = 35 \) seconds. Therefore, at \( t = 35 \) seconds, runner 1 is only \( 8 + (-2) = 6 \) feet ahead.

49. Express the area (not signed) of the shaded region in Figure 16 as a sum of three integrals involving the functions \( f \) and \( g \).

![Figure 16](image)

SOLUTION Because either the curve bounding the top of the region or the curve bounding the bottom of the region or both change at \( x = 3 \) and at \( x = 5 \), the area is calculated using three integrals. Specifically, the area is

\[
\int_0^3 (f(x) - g(x)) \, dx + \int_3^5 (f(x) - 0) \, dx + \int_5^9 (0 - f(x)) \, dx \\
= \int_0^3 (f(x) - g(x)) \, dx + \int_3^5 f(x) \, dx - \int_5^9 f(x) \, dx.
\]

50. Find the area enclosed by the curves \( y = c - x^2 \) and \( y = x^2 - c \) as a function of \( c \). Find the value of \( c \) for which this area is equal to 1.
**SOLUTION** The curves intersect at \( x = \pm \sqrt{c} \), with \( y = c - x^2 \) above \( y = x^2 - c \) over the interval \([-\sqrt{c}, \sqrt{c}]\). The area of the region enclosed by the two curves is then

\[
\int_{-\sqrt{c}}^{\sqrt{c}} \left( c - x^2 \right) - \left( x^2 - c \right) \, dx = \int_{-\sqrt{c}}^{\sqrt{c}} \left( 2c - 2x^2 \right) \, dx = \left( 2cx - \frac{2}{3} x^3 \right) \bigg|_{-\sqrt{c}}^{\sqrt{c}} = \frac{8}{3} c^{3/2}.
\]

In order for the area to equal 1, we must have \( \frac{8}{3} c^{3/2} = 1 \), which gives

\[
c = \frac{9^{1/3}}{4} \approx 0.520021.
\]

**51.** Set up (but do not evaluate) an integral that expresses the area between the circles \( x^2 + y^2 = 2 \) and \( x^2 + (y-1)^2 = 1 \).

**SOLUTION** Setting \( 2 - y^2 = 1 - (y-1)^2 \) yields \( y = 1 \). The two circles therefore intersect at the points \((1, 1)\) and \((-1, 1)\). From the graph below, we see that over the interval \([-1, 1]\), the upper half of the circle \( x^2 + y^2 = 2 \) lies above the lower half of the circle \( x^2 + (y-1)^2 = 1 \). The area enclosed by the two circles is therefore given by the integral

\[
\int_{-1}^{1} \left( \sqrt{2 - x^2} - \left(1 - \sqrt{1 - x^2}\right) \right) \, dx.
\]

**52.** Set up (but do not evaluate) an integral that expresses the area between the graphs of \( y = (1 + x^2)^{-1} \) and \( y = x^2 \).

**SOLUTION** Setting \( (1 + x^2)^{-1} = x^2 \) yields \( x^4 + x^2 - 1 = 0 \). This is a quadratic equation in the variable \( x^2 \). By the quadratic formula,

\[
x^2 = \frac{-1 \pm \sqrt{1 - 4(-1)}}{2} = \frac{-1 \pm \sqrt{5}}{2}.
\]

As \( x^2 \) must be nonnegative, we discard \( \frac{-1 - \sqrt{5}}{2} \). Finally, we find the two curves intersect at \( x = \pm \sqrt{\frac{1 + \sqrt{5}}{2}} \). From the graph below, we see that \( y = (1 + x^2)^{-1} \) lies above \( y = x^2 \). The area enclosed by the two curves is then

\[
\int_{\sqrt{\frac{1 - \sqrt{5}}{2}}}^{\sqrt{\frac{1 + \sqrt{5}}{2}}} \left( (1 + x^2)^{-1} - x^2 \right) \, dx.
\]

**53.** **CAS** Find a numerical approximation to the area above \( y = 1 - (x/\pi) \) and below \( y = \sin x \) (find the points of intersection numerically).

**SOLUTION** The region in question is shown in the figure below. Using a computer algebra system, we find that \( y = 1 - x/\pi \) and \( y = \sin x \) intersect on the left at \( x = 0.8278585215 \). Analytically, we determine the two curves intersect on the right at \( x = \pi \). The area above \( y = 1 - x/\pi \) and below \( y = \sin x \) is then

\[
\int_{0.8278585215}^{\pi} \left( \sin x - \left(1 - \frac{x}{\pi}\right) \right) \, dx = 0.8244398727,
\]

where the definite integral was evaluated using a computer algebra system.
54. **CAS** Find a numerical approximation to the area above \( y = |x| \) and below \( y = \cos x \).

**SOLUTION** The region in question is shown in the figure below. We see that the region is symmetric with respect to the \( y \)-axis, so we can determine the total area of the region by doubling the area of the portion in the first quadrant. Using a computer algebra system, we find that \( y = \cos x \) and \( y = |x| \) intersect at \( x = 0.7390851332 \). The area of the region between the two curves is then 

\[
2 \int_0^{0.7390851332} (\cos x - x) \, dx = 0.8009772242,
\]

where the definite integral was evaluated using a computer algebra system.

55. **CAS** Use a computer algebra system to find a numerical approximation to the number \( c \) (besides zero) in \([0, \pi/2]\), where the curves \( y = \sin x \) and \( y = \tan^2 x \) intersect. Then find the area enclosed by the graphs over \([0, c]\).

**SOLUTION** The region in question is shown in the figure below. Using a computer algebra system, we find that \( y = \sin x \) and \( y = \tan^2 x \) intersect at \( x = 0.6662394325 \). The area of the region enclosed by the two curves is then 

\[
\int_0^{0.6662394325} (\sin x - \tan^2 x) \, dx = 0.09393667698,
\]

where the definite integral was evaluated using a computer algebra system.

56. The back of Jon’s guitar (Figure 17) has a length 19 in. He measured the widths at 1-in. intervals, beginning and ending \( \frac{1}{2} \) in. from the ends, obtaining the results

\[6, 9, 10.25, 10.75, 10.75, 10.25, 9.75, 9.5, 10, 11.25, 12.75, 13.75, 14.25, 14.5, 14.5, 14, 13.25, 11.25, 9\]

Use the midpoint rule to estimate the area of the back.

**FIGURE 17** Back of guitar.
SOLUTION  Note that the measurements were taken at the midpoint of each one-inch section of the guitar. For example, in the 0 to 1 inch section, the midpoint would be at $\frac{1}{2}$ inch, and thus the approximate area of the first rectangle would be $1 \cdot 6$ inches$^2$. An approximation for the entire area is then

$$A = 1(6 + 9 + 10.25 + 10.75 + 10.75 + 10.25 + 9.75 + 9.5 + 10 + 11.25$$

$$+ 12.75 + 13.75 + 14.25 + 14.5 + 14 + 13.25 + 11.25 + 9)$$

$$= 214.75 \text{ in}^2.$$ 

Further Insights and Challenges

57. Find the line $y = mx$ that divides the area under the curve $y = x(1 - x)$ over $[0, 1]$ into two regions of equal area.

SOLUTION  First note that

$$\int_0^1 x(1 - x) \, dx = \int_0^1 (x - x^2) \, dx = \left[ \frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_0^1 = \frac{1}{6}.$$ 

Now, the line $y = mx$ and the curve $y = x(1 - x)$ intersect when $mx = x(1 - x)$, or at $x = 0$ and at $x = 1 - m$. The area of the region enclosed by the two curves is then

$$\int_0^{1-m} (x(1 - x) - mx) \, dx = \int_0^{1-m} ((1 - m)x - x^2) \, dx = \left[ (1 - m)x^2 - \frac{1}{3}x^3 \right]_0^{1-m} = \frac{1}{6}(1 - m)^3.$$ 

To have $\frac{1}{6}(1 - m)^3 = \frac{1}{2} \cdot \frac{1}{6}$ requires

$$m = 1 - \left( \frac{1}{2} \right)^{1/3} \approx 0.206299.$$ 

58. Let $c$ be the number such that the area under $y = \sin x$ over $[0, \pi]$ is divided by the line $y = cx$ (Figure 18). Find an equation for $c$ and solve this equation numerically using a computer algebra system.

SOLUTION  First note that

$$\int_0^\pi \sin x \, dx = -\cos x \bigg|_0^\pi = 2.$$ 

Now, let $y = cx$ and $y = \sin x$ intersect at $x = a$. Then $ca = \sin a$, which gives $c = \frac{\sin a}{a}$ and $y = cx = \frac{\sin a}{a}x$. Then

$$\int_0^a \left( \sin x - \frac{\sin a}{a}x \right) \, dx = \left[ -\cos x - \frac{\sin a}{2a}x^2 \right]_0^a = 1 - \cos a - \frac{a \sin a}{2}.$$ 

We need

$$1 - \cos a - \frac{a \sin a}{2} = \frac{1}{2} \cdot (2) = 1,$$

which gives $a = 2.458714176$ and finally

$$c = \frac{\sin a}{a} = 0.256649857.$$ 

59. Explain geometrically (without calculation) why the following holds for any $n > 0$:

$$\int_0^1 x^n \, dx + \int_0^1 x^{1/n} \, dx = 1.$$
SOLUTION Let $A_1$ denote the area of region 1 in the figure below. Define $A_2$ and $A_3$ similarly. It is clear from the figure that

$$A_1 + A_2 + A_3 = 1.$$ 

Now, note that $x^n$ and $x^{1/n}$ are inverses of each other. Therefore, the graphs of $y = x^n$ and $y = x^{1/n}$ are symmetric about the line $y = x$, so regions 1 and 3 are also symmetric about $y = x$. This guarantees that $A_1 = A_3$. Finally,

$$\int_0^1 x^n \, dx + \int_0^1 x^{1/n} \, dx = A_3 + (A_2 + A_3) = A_1 + A_2 + A_3 = 1.$$ 

To solve the problem, let $f(x)$ be a strictly increasing function with inverse $g(x)$. Explain the equality geometrically:

$$\int_0^a f(x) \, dx + \int_{f(0)}^{f(a)} g(x) \, dx = af(a).$$

**SOLUTION** The region whose area is represented by $\int_0^a f(x) \, dx$ is shown as the shaded portion of the graph below on the left, and the region whose area is represented by $\int_{f(0)}^{f(a)} g(x) \, dx$ is shown as the shaded portion of the graph below on the right. Because $f$ and $g$ are inverse functions, the graph of $y = f(x)$ is obtained by reflecting the graph of $y = g(x)$ through the line $y = x$. It then follows that if we were to reflect the shaded region in the graph below on the right through the line $y = x$, the reflected region would coincide exactly with the region $R$ in the graph below on the left. Thus

$$\int_0^a f(x) \, dx + \int_{f(0)}^{f(a)} g(x) \, dx = \text{area of a rectangle with width } a \text{ and height } f(a) = af(a).$$

### 6.2 Setting Up Integrals: Volume, Density, Average Value

**Preliminary Questions**

1. What is the average value of $f(x)$ on $[1, 4]$ if the area between the graph of $f(x)$ and the $x$-axis is equal to 9?

**SOLUTION** Assuming that $f(x) \geq 0$ over the interval $[1, 4]$, the fact that the area between the graph of $f$ and the $x$-axis is equal to 9 indicates that $\int_1^4 f(x) \, dx = 9$. The average value of $f$ over the interval $[1, 4]$ is then

$$\frac{\int_1^4 f(x) \, dx}{4-1} = \frac{9}{3} = 3.$$ 

2. Find the volume of a solid extending from $y = 2$ to $y = 5$ if the cross section at $y$ has area $A(y) = 5$ for all $y$.

**SOLUTION** Because the cross-sectional area of the solid is constant, the volume is simply the cross-sectional area times the length, or $5 \times 3 = 15$.

3. Describe the horizontal cross sections of an ice cream cone and the vertical cross sections of a football (when it is held horizontally).
SECTION 6.2 | Setting Up Integrals: Volume, Density, Average Value

**SOLUTION**  The horizontal cross sections of an ice cream cone, as well as the vertical cross sections of a football (when held horizontally), are circles.

4. What is the formula for the total population within a circle of radius \( R \) around a city center if the population has a radial function?

**SOLUTION**  Because the population density is a radial function, the total population within a circle of radius \( R \) is

\[ 2\pi \int_0^R r\rho(r) \, dr, \]

where \( \rho(r) \) is the radial population density function.

5. What is the definition of flow rate?

**SOLUTION**  The flow rate of a fluid is the volume of fluid that passes through a cross-sectional area at a given point per unit time.

6. Which assumption about fluid velocity did we use to compute the flow rate as an integral?

**SOLUTION**  To express flow rate as an integral, we assumed that the fluid velocity depended only on the radial distance from the center of the tube.

**Exercises**

1. Let \( V \) be the volume of a pyramid of height 20 whose base is a square of side 8.

(a) Use similar triangles as in Example 1 to find the area of the horizontal cross section at a height \( y \).

(b) Calculate \( V \) by integrating the cross-sectional area.

**SOLUTION**

(a) We can use similar triangles to determine the side length, \( s \), of the square cross section at height \( y \). Using the diagram below, we find

\[ \frac{8}{20} = \frac{s}{20 - y} \text{ or } s = \frac{2}{5}(20 - y). \]

The area of the cross section at height \( y \) is then given by \( \frac{4}{25}(20 - y)^2 \).

(b) The volume of the pyramid is

\[ \int_0^{20} \frac{4}{25}(20 - y)^2 \, dy = -\frac{4}{75}(20 - y)^3 \bigg|_0^{20} = \frac{1280}{3}. \]

2. Let \( V \) be the volume of a right circular cone of height 10 whose base is a circle of radius 4 (Figure 16).

**FIGURE 16** Right circular cones.

(a) Use similar triangles to find the area of a horizontal cross section at a height \( y \).

(b) Calculate \( V \) by integrating the cross-sectional area.

**SOLUTION**
(a) If \( r \) is the radius at height \( y \) (see Figure 16), then
\[
\frac{10}{4} = \frac{10 - y}{r}
\]
from similar triangles, which implies that \( r = 4 - \frac{2}{5}y \). The area of the cross-section at height \( y \) is then
\[
A = \pi \left( 4 - \frac{2}{5}y \right)^2.
\]
(b) The volume of the cone is
\[
V = \int_0^{10} \pi \left( 4 - \frac{2}{5}y \right)^2 \, dy = \frac{5\pi}{6} \left| \left( 4 - \frac{2}{5}y \right)^3 \right|_0^{10} = \frac{160\pi}{3}.
\]

3. Use the method of Exercise 2 to find the formula for the volume of a right circular cone of height \( h \) whose base is a circle of radius \( r \) (Figure 16).

**SOLUTION**

(a) From similar triangles (see Figure 16),
\[
\frac{h}{h - y} = \frac{r}{r_0},
\]
where \( r_0 \) is the radius of the cone at a height of \( y \). Thus, \( r_0 = r - \frac{ry}{h} \).

(b) The volume of the cone is
\[
\pi \int_0^h \left( r - \frac{ry}{h} \right)^2 \, dy = \frac{-h\pi}{r} \left( r - \frac{ry}{h} \right)^3 \bigg|_0^h = \frac{\pi r^3}{3} = \frac{\pi rh^3}{3}.
\]

4. Calculate the volume of the ramp in Figure 17 in three ways by integrating the area of the cross sections:

(a) Perpendicular to the \( x \)-axis (rectangles)

(b) Perpendicular to the \( y \)-axis (triangles)

(c) Perpendicular to the \( z \)-axis (rectangles)

![Figure 17](image-url)  
**FIGURE 17** Ramp of length 6, width 4, and height 2.

**SOLUTION**

(a) Cross sections perpendicular to the \( x \)-axis are rectangles of width 4 and height \( 2 - \frac{1}{3}x \). The volume of the ramp is then
\[
\int_0^6 4 \left( -\frac{1}{3}x + 2 \right) \, dx = \left( -\frac{2}{3}x^2 + 8x \right) \bigg|_0^6 = 24.
\]

(b) Cross sections perpendicular to the \( y \)-axis are right triangles with legs of length 2 and 6. The volume of the ramp is then
\[
\int_0^4 \left( \frac{1}{2} \cdot 2 \cdot 6 \right) \, dy = (6y) \bigg|_0^4 = 24.
\]

(c) Cross sections perpendicular to the \( z \)-axis are rectangles of length \( 6 - 3z \) and width 4. The volume of the ramp is then
\[
\int_0^2 4 \left( -3(z - 2) \right) \, dz = (-6z^2 + 24z) \bigg|_0^2 = 24.
\]
5. Find the volume of liquid needed to fill a sphere of radius $R$ to height $h$ (Figure 18).

![Figure 18](sphere_filled_with_liquid_to_height_h.png)

**SOLUTION** The radius $r$ at any height $y$ is given by $r = \sqrt{R^2 - (R - y)^2}$. Thus, the volume of the filled portion of the sphere is

$$\pi \int_0^h r^2 \, dy = \pi \int_0^h \left( R^2 - (R - y)^2 \right) \, dy = \pi \left( Ry^2 - \frac{y^3}{3} \right) \bigg|_0^h = \pi \left( Rh^2 - \frac{h^3}{3} \right).$$

6. Find the volume of the wedge in Figure 19(A) by integrating the area of vertical cross sections.

![Figure 19](wedge_cross_sections.png)

**SOLUTION** Cross sections of the wedge taken perpendicular to the $x$-axis are right triangles. Using similar triangles, we find the base and the height of the cross sections to be $\frac{3}{4}(8 - x)$ and $\frac{1}{2}(8 - x)$, respectively. The volume of the wedge is then

$$\frac{3}{16} \int_0^8 (8 - x)^2 \, dx = \frac{3}{16} \int_0^8 \left( 64 - 16x + x^2 \right) \, dx = \frac{3}{16} \left( 64x - 8x^2 + \frac{1}{3}x^3 \right) \bigg|_0^8 = 32.$$

7. Derive a formula for the volume of the wedge in Figure 19(B) in terms of the constants $a$, $b$, and $c$.

**SOLUTION** The line from $c$ to $a$ is given by the equation $(z/c) + (x/a) = 1$ and the line from $b$ to $a$ is given by $(y/b) + (x/a) = 1$. The cross sections perpendicular to the $x$-axis are right triangles with height $c(1 - x/a)$ and base $b(1 - x/a)$. Thus we have

$$\int_0^a \frac{1}{2} bc \left( 1 - \frac{x}{a} \right)^2 \, dx = -\frac{1}{6} abc \left( 1 - \frac{x}{a} \right) \bigg|_0^a = \frac{1}{6} abc.$$

8. Let $B$ be the solid whose base is the unit circle $x^2 + y^2 = 1$ and whose vertical cross sections perpendicular to the $x$-axis are equilateral triangles. Show that the vertical cross sections have area $A(x) = \sqrt{3}(1 - x^2)$ and compute the volume of $B$.

**SOLUTION** At the arbitrary location $x$, the side of the equilateral triangle cross section that lies in the base of the solid extends from the top half of the unit circle (with $y = \sqrt{1 - x^2}$) to the bottom half (with $y = -\sqrt{1 - x^2}$). The equilateral triangle therefore has sides of length $s = 2\sqrt{1 - x^2}$ and an area of

$$A(x) = \frac{s^2\sqrt{3}}{4} = \sqrt{3}(1 - x^2).$$

Finally, the volume of the solid is

$$\sqrt{3} \int_{-1}^1 \left( 1 - x^2 \right) \, dx = \sqrt{3} \left( x - \frac{1}{3}x^3 \right) \bigg|_{-1}^1 = \frac{4\sqrt{3}}{3}.$$

In Exercises 9–14, find the volume of the solid with given base and cross sections.

9. The base is the unit circle $x^2 + y^2 = 1$ and the cross sections perpendicular to the $x$-axis are triangles whose height and base are equal.
10. The base is the triangle enclosed by $x + y = 1$, the $x$-axis, and the $y$-axis. The cross sections perpendicular to the $y$-axis are semicircles.

**SOLUTION** The diameter of the semicircle lies in the base of the solid and thus has length $1 - y$ for each $y$. The area of the semicircle is then

$$\frac{1}{2} \pi \left( \frac{1 - y}{2} \right)^2 = \frac{1}{8} \pi (1 - y)^2.$$

Finally, the volume of the solid is

$$\frac{\pi}{8} \int_0^1 (1 - y)^2 \, dy = \frac{\pi}{8} \int_0^1 (1 - 2y + y^2) \, dy = \frac{\pi}{8} \left( y - y^2 + \frac{1}{3} y^3 \right) \bigg|_0^1 = \frac{\pi}{24}.$$

11. The base is the semicircle $y = \sqrt{9 - x^2}$, where $-3 \leq x \leq 3$. The cross sections perpendicular to the $x$-axis are squares.

**SOLUTION** For each $x$, the base of the square cross section extends from the semicircle $y = \sqrt{9 - x^2}$ to the $x$-axis. The square therefore has a base with length $\sqrt{9 - x^2}$ and an area of $\left( \sqrt{9 - x^2} \right)^2 = 9 - x^2$. The volume of the solid is then

$$\int_{-3}^{3} (9 - x^2) \, dx = \left( 9x - \frac{1}{3} x^3 \right) \bigg|_{-3}^{3} = 36.$$

12. The base is a square, one of whose sides is the interval $[0, \ell]$ along the $x$-axis. The cross sections perpendicular to the $x$-axis are rectangles of height $f(x) = x^2$.

**SOLUTION** For each $x$, the rectangular cross section has base $\ell$ and height $x^2$. The cross-sectional area is then $\ell x^2$, and the volume of the solid is

$$\int_0^\ell (\ell x^2) \, dx = \left( \frac{1}{3} \ell x^3 \right) \bigg|_0^\ell = \frac{1}{3} \ell^4.$$

13. The base is the region enclosed by $y = x^2$ and $y = 3$. The cross sections perpendicular to the $y$-axis are squares.

**SOLUTION** At any location $y$, the distance to the parabola from the $y$-axis is $\sqrt{y}$. Thus the base of the square will have length $2\sqrt{y}$. Therefore the volume is

$$\int_0^3 \left(2\sqrt{y}\right)^2 2\sqrt{y} \, dy = \int_0^3 4y^2 \, dy = 2y^3 \bigg|_0^3 = 18.$$

14. The base is the region enclosed by $y = x^2$ and $y = 3$. The cross sections perpendicular to the $y$-axis are rectangles of height $y^3$.

**SOLUTION** As in previous exercise, for each $y$, the width of the rectangle will be $2\sqrt{y}$. Because the height is $y^3$, the volume of the solid is given by

$$2 \int_0^3 y^{7/2} \, dy = \frac{4}{9} y^{9/2} \bigg|_0^3 = 36\sqrt{3}.$$

15. Find the volume of the solid whose base is the region $|x| + |y| \leq 1$ and whose vertical cross sections perpendicular to the $y$-axis are semicircles (with diameter along the base).

**SOLUTION** The region $R$ in question is a diamond shape connecting the points $(1, 0)$, $(0, -1)$, $(-1, 0)$, and $(0, 1)$. Thus, in the lower half of the $xy$-plane, the radius of the circles is $y + 1$ and in the upper half, the radius is $1 - y$. Therefore, the volume is

$$\frac{\pi}{2} \int_{-1}^{0} (y + 1)^2 \, dy + \frac{\pi}{2} \int_{0}^{1} (1 - y)^2 \, dy = \frac{\pi}{2} \left( \frac{1}{3} + \frac{1}{3} \right) = \frac{\pi}{3}.$$
16. Show that the volume of a pyramid of height \( h \) whose base is an equilateral triangle of side \( s \) is equal to \( \frac{\sqrt{3}}{12} hs^2 \).

**SOLUTION** Using similar triangles, the side length of the equilateral triangle at height \( x \) above the base is \( \frac{s(h-x)}{h} \); the area of the cross section is therefore given by \( \frac{\sqrt{3}}{4} \left( \frac{s(h-x)}{h} \right)^2 \).

Thus, the volume of the pyramid is

\[
\frac{s^2 \sqrt{3}}{4h^2} \int_0^h (h-x)^2 \, dx = \left( -\frac{s^2 \sqrt{3}}{12h^2} (h-x)^3 \right)_0^h = \frac{\sqrt{3}}{12} s^2 h.
\]

17. Find the volume \( V \) of a regular tetrahedron whose face is an equilateral triangle of side \( s \) (Figure 20).

![Regular tetrahedron](image)

**FIGURE 20** Regular tetrahedron.

**SOLUTION** Our first task is to determine the relationship between the height of the tetrahedron, \( h \), and the side length of the equilateral triangles, \( s \). Let \( B \) be the orthocenter of the tetrahedron (the point directly below the apex), and let \( b \) denote the distance from \( B \) to each corner of the base triangle. By the Law of Cosines, we have

\[ s^2 = b^2 + b^2 - 2b^2 \cos 120^\circ = 3b^2, \]

so \( b^2 = \frac{1}{3}s^2 \). Thus

\[ h^2 = s^2 - b^2 = \frac{2}{3}s^2 \quad \text{or} \quad h = s\sqrt{\frac{2}{3}}. \]

Therefore, using similar triangles, the side length of the equilateral triangle at height \( z \) above the base is

\[ s \left( \frac{h-z}{h} \right) = s - \frac{z}{\sqrt{2/3}}. \]

The volume of the tetrahedron is then given by

\[
\int_0^{s\sqrt{2/3}} \frac{\sqrt{3}}{4} \left( s - \frac{z}{\sqrt{2/3}} \right)^2 \, dz = \left. -\frac{\sqrt{2}}{12} \left( s - \frac{z}{\sqrt{2/3}} \right)^3 \right|_0^{s\sqrt{2/3}} = \frac{s^3 \sqrt{2}}{12}.
\]

18. The area of an ellipse is \( \pi ab \), where \( a \) and \( b \) are the lengths of the semimajor and semiminor axes (Figure 21). Compute the volume of a cone of height 12 whose base is an ellipse with semimajor axis \( a = 6 \) and semiminor axis \( b = 4 \).

![Cone with elliptical base](image)

**FIGURE 21**

**SOLUTION** At each height \( y \), the elliptical cross section has major axis \( \frac{1}{2}(12 - y) \) and minor axis \( \frac{1}{2}(12 - y) \). The cross-sectional area is then \( \frac{\pi}{6}(12 - y)^2 \), and the volume is

\[
\int_0^{12} \frac{\pi}{6} (12 - y)^2 \, dy = \left. -\frac{\pi}{18} (12 - y)^3 \right|_0^{12} = 96\pi.
\]
19. A frustum of a pyramid is a pyramid with its top cut off [Figure 22(A)]. Let \( V \) be the volume of a frustum of height \( h \) whose base is a square of side \( a \) and top is a square of side \( b \) with \( a > b \geq 0 \).

(a) Show that if the frustum were continued to a full pyramid, it would have height \( \frac{ha}{a-b} \) [Figure 22(B)].

(b) Show that the cross section at height \( x \) is a square of side \( \left( \frac{1}{h} \right) \left( a \left( h-x \right) + bx \right) \).

(c) Show that \( V = \frac{1}{3}h(a^2 + ab + b^2) \). A papyrus dating to the year 1850 BCE indicates that Egyptian mathematicians had discovered this formula almost 4,000 years ago.

![Figure 22](image)

**SOLUTION**

(a) Let \( H \) be the height of the full pyramid. Using similar triangles, we have the proportion

\[
\frac{H}{a} = \frac{H-h}{b}
\]

which gives

\[
H = \frac{ha}{a-b}.
\]

(b) Let \( w \) denote the side length of the square cross section at height \( x \). By similar triangles, we have

\[
\frac{a}{H} = \frac{w}{H-x}.
\]

Substituting the value for \( H \) from part (a) gives

\[
w = \frac{a(h-x) + bx}{h}.
\]

(c) The volume of the frustum is

\[
\int_0^h \left( \frac{1}{h} \left( a(h-x) + bx \right) \right)^2 dx = \frac{1}{h^2} \int_0^h \left( a^2(h-x)^2 + 2ab(h-x)x + b^2x^2 \right) dx
\]

\[
= \frac{1}{h^2} \left( -\frac{a^2}{3} (h-x)^3 + abhx^2 - \frac{2}{3} abx^3 + \frac{1}{3} b^2x^3 \right) \bigg|_0^h = \frac{h}{3} \left( a^2 + ab + b^2 \right).
\]

20. A plane inclined at an angle of 45° passes through a diameter of the base of a cylinder of radius \( r \). Find the volume of the region within the cylinder and below the plane (Figure 23).

![Figure 23](image)

**SOLUTION** Place the center of the base at the origin. Then, for each \( x \), the vertical cross section taken perpendicular to the \( x \)-axis is a rectangle of base \( 2\sqrt{r^2 - x^2} \) and height \( x \). The volume of the solid enclosed by the plane and the cylinder is therefore

\[
\int_0^r 2x\sqrt{r^2 - x^2} dx = \int_0^{r^2} \sqrt{a} du = \left( \frac{2}{3} a^{3/2} \right) \bigg|_0^{r^2} = \frac{2}{3} r^3.
\]

21. Figure 24 shows the solid \( S \) obtained by intersecting two cylinders of radius \( r \) whose axes are perpendicular.
(a) The horizontal cross section of each cylinder at distance \( y \) from the central axis is a rectangular strip. Find the strip’s width.

(b) Find the area of the horizontal cross section of \( S \) at distance \( y \).

(c) Find the volume of \( S \) as a function of \( r \).

**FIGURE 24** Intersection of two cylinders intersecting at right angles.

**SOLUTION**

(a) The horizontal cross section at distance \( y \) from the central axis (for \(-r \leq y \leq r\)) is a square of width \( w = 2\sqrt{r^2 - y^2} \).

(b) The area of the horizontal cross section of \( S \) at distance \( y \) from the central axis is \( w^2 = 4(r^2 - y^2) \).

(c) The volume of the solid \( S \) is then

\[ 4 \int_{-r}^{r} (r^2 - y^2) \, dy = 4 \left( r^2 y - \frac{1}{3} y^3 \right) \Bigg|_{-r}^{r} = \frac{16}{3} r^3. \]

22. Let \( S \) be the solid obtained by intersecting two cylinders of radius \( r \) whose axes intersect at an angle \( \theta \). Find the volume of \( S \) as a function of \( r \) and \( \theta \).

**SOLUTION** Each cross section at distance \( y \) from the central axis (for \(-r \leq y \leq r\)) is a rhombus with side length \( \frac{2\sqrt{r^2 - y^2}}{\sin \theta} \). The area of each rhombus is \( \frac{4(r^2 - y^2)}{\sin \theta} \), and thus the volume of the solid will be

\[ \frac{4}{\sin \theta} \int_{-r}^{r} (r^2 - y^2) \, dy = \frac{16r^3}{3 \sin \theta}. \]

23. Calculate the volume of a cylinder inclined at an angle \( \theta = 30^\circ \) whose height is 10 and whose base is a circle of radius 4 (Figure 25).

**FIGURE 25** Cylinder inclined at an angle \( \theta = 30^\circ \).

**SOLUTION** The area of each circular cross section is \( \pi(4)^2 = 16\pi \), hence the volume of the cylinder is

\[ \int_{0}^{10} 16\pi \, dx = (16\pi x) \bigg|_{0}^{10} = 160\pi \]

24. Find the total mass of a 1-m rod whose linear density function is \( \rho(x) = 10(x + 1)^{-2} \) kg/m for \( 0 \leq x \leq 1 \).

**SOLUTION** The total mass of the rod is

\[ \int_{0}^{1} \rho(x) \, dx = \int_{0}^{1} \left( 10(x + 1)^{-2} \right) \, dx = \left( -10(x + 1)^{-1} \right) \bigg|_{0}^{1} = 5 \text{ kg}. \]

25. Find the total mass of a 2-m rod whose linear density function is \( \rho(x) = 1 + 0.5 \sin(x) \) kg/m for \( 0 \leq x \leq 2 \).

**SOLUTION** The total mass of the rod is

\[ \int_{0}^{2} \rho(x) \, dx = \int_{0}^{2} (1 + 0.5 \sin \pi x) \, dx = \left( x - \frac{0.5 \cos \pi x}{\pi} \right) \bigg|_{0}^{2} = 2 \text{ kg}. \]
26. A mineral deposit along a strip of length 6 cm has density $s(x) = 0.01x(6-x)$ g/cm for $0 \leq x \leq 6$. Calculate the total mass of the deposit.

**SOLUTION** The total mass of the deposit is

$$\int_0^6 s(x) \, dx = \int_0^6 0.01x(6-x) \, dx = \left(0.03x^2 - \frac{0.01}{3}x^3\right)_0^6 = 0.36 \text{ g.}$$

27. Calculate the population within a 10-mile radius of the city center if the radial population density is $\rho(r) = 4(1 + r^2)^{1/3}$ (in thousands per square mile).

**SOLUTION** The total population is

$$2\pi \int_0^{10} r \cdot \rho(r) \, dr = 2\pi \int_0^{10} 4r(1 + r^2)^{1/3} \, dr = 3\pi(1 + r^2)^{4/3}\bigg|_0^{10} \approx 4423.59 \text{ thousand} \approx 4.4 \text{ million.}$$

28. Odzala National Park in the Congo has a high density of gorillas. Suppose that the radial population density is $\rho(r) = 52(1 + r^2)^{-5/2}$ gorillas per square kilometer, where $r$ is the distance from a large grassy clearing with a source of food and water. Calculate the number of gorillas within a 5-km radius of the clearing.

**SOLUTION** The number of gorillas within a 5-km radius of the clearing is

$$2\pi \int_0^5 r \cdot \rho(r) \, dr = \int_0^5 \frac{104\pi r}{(1 + r^2)^{3/2}} \, dr = \frac{52\pi}{1 + r^2}\bigg|_0^5 = 50\pi \approx 157.$$ 

29. Table 1 lists the population density (in people per squared kilometer) as a function of distance $r$ (in kilometers) from the center of a rural town. Estimate the total population within a 2-km radius of the center by taking the average of the left- and right-endpoint approximations.

<table>
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<td>37.6</td>
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<tr>
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<td>46.0</td>
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</table>

**SOLUTION** The total population is given by

$$2\pi \int_0^2 r \cdot \rho(r) \, dr.$$

With $\Delta r = 0.2$, the left- and right-endpoint approximations to the required definite integral are

$$L_{10} = 0.2(2\pi)(0)(125) + (0.2)(102.3) + (0.4)(83.8) + (0.6)(68.6) + (0.8)(56.2) + (1)(46)$$

$$+ (1.2)(37.6) + (1.4)(30.8) + (1.6)(25.2) + (1.8)(20.7) = 442.24;$$

$$R_{10} = 0.2(2\pi)(102.3) + (0.4)(83.8) + (0.6)(68.6) + (0.8)(56.2) + (1)(46)$$

$$+ (1.2)(37.6) + (1.4)(30.8) + (1.6)(25.2) + (1.8)(20.7) + (2)(16.9) = 484.71.$$ 

This gives an average of 463.475. Thus, there are roughly 463 people within a 2-km radius of the town center.

30. Find the total mass of a circular plate of radius 20 cm whose mass density is the radial function $\rho(r) = 0.03 + 0.01 \cos(\pi r^2)$ g/cm².

**SOLUTION** The total mass of the plate is

$$2\pi \int_0^{20} r \cdot \rho(r) \, dr = 2\pi \int_0^{20} (0.03 + 0.01 \cos(\pi r^2)) \, dr = 2\pi \left(0.015r^2 + \frac{0.01}{2\pi} \sin(\pi r^2)\right)_0^{20} = 12\pi \text{ grams.}$$
31. The density of deer in a forest is the radial function \( \rho(r) = 150(r^2 + 2)^{-2} \) deer per km\(^2\), where \( r \) is the distance (in kilometers) to a small meadow. Calculate the number of deer in the region \( 2 \leq r \leq 5 \) km.

**SOLUTION** The number of deer in the region \( 2 \leq r \leq 5 \) km is

\[
2\pi \int_2^5 r \left( \frac{150}{(r^2 + 2)^2} \right) \, dr = -150\pi \left( \frac{1}{27} - \frac{1}{6} \right) \approx 61 \text{ deer}.
\]

32. Show that a circular plate of radius 2 cm with radial mass density \( \rho(r) = \frac{4}{r} \text{ g/cm} \) has finite total mass, even though the density becomes infinite at the origin.

**SOLUTION** The total mass of the plate is

\[
2\pi \int_0^2 \left( \frac{4}{r} \right) \, dr = 16\pi \text{ g}.
\]

33. Find the flow rate through a tube of radius 4 cm, assuming that the velocity of fluid particles at a distance \( r \) cm from the center is \( v(r) = 16 - r^2 \) cm/s.

**SOLUTION** The flow rate is

\[
2\pi \int_0^R r v(r) \, dr = 2\pi \int_0^4 \left( 16 - r^2 \right) \, dr = 128\pi \text{ cm}^3/\text{s}.
\]

34. Let \( v(r) \) be the velocity of blood in an arterial capillary of radius \( R = 4 \times 10^{-5} \) m. Use Poiseuille’s Law (Example 6) with \( k = 10^6 \) (m-s)\(^{-1}\) to determine the velocity at the center of the capillary and the flow rate (use correct units).

**SOLUTION** According to Poiseuille’s Law, \( v(r) = k(R^2 - r^2) \). With \( R = 4 \times 10^{-5} \) m and \( k = 10^6 \) (m-s)\(^{-1}\),

\[
v(0) = 0.0016 \text{ m/s}.
\]

The flow rate through the capillary is

\[
2\pi \int_0^R kr(R^2 - r^2) \, dr = 2\pi k \left( \frac{R^2 r^2}{2} - \frac{r^4}{4} \right) \bigg|_0^R = 4.02 \times 10^{-12} \text{ m}^3/\text{s}.
\]

35. A solid rod of radius 1 cm is placed in a pipe of radius 3 cm so that their axes are aligned. Water flows through the pipe and around the rod. Find the flow rate if the velocity of the water is given by the radial function \( v(r) = 0.5(r - 1)(3 - r) \) cm/s.

**SOLUTION** The flow rate is

\[
2\pi \int_1^3 r (0.5)(r - 1)(3 - r) \, dr = \frac{8\pi}{3} \text{ cm}^3/\text{s}.
\]

36. To estimate the volume \( V \) of Lake Nogebow, the Minnesota Bureau of Fisheries created the depth contour map in Figure 26 and determined the area of the cross section of the lake at the depths recorded in the table below. Estimate \( V \) by taking the average of the right- and left-endpoint approximations to the integral of cross-sectional area.

<table>
<thead>
<tr>
<th>Depth (ft)</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Area (million ft(^2))</td>
<td>2.1</td>
<td>1.5</td>
<td>1.1</td>
<td>0.835</td>
<td>0.217</td>
</tr>
</tbody>
</table>
SOLUTION  The volume of the lake is
\[ \int_{0}^{20} A(z) \, dz, \]
where \( A(z) \) denotes the cross-sectional area of the lake at depth \( z \). The right- and left-endpoint approximations to this integral, with \( \Delta z = 5 \), are
\[ R = 5 (1.5 + 1.1 + .835 + .217) = 18.26 \]
\[ L = 5 (2.1 + 1.5 + 1.1 + .835) = 27.675 \]
Thus
\[ V \approx \frac{1}{2} (18.26 + 27.675) = 22.97 \text{ million ft}^3. \]

In Exercises 37–46, calculate the average over the given interval.

37. \( f(x) = x^3, \quad [0, 1] \)
SOLUTION  The average is
\[ \frac{1}{1-0} \int_{0}^{1} x^3 \, dx = \frac{1}{4} x^4 \bigg|_{0}^{1} = \frac{1}{4}. \]

38. \( f(x) = x^3, \quad [-1, 1] \)
SOLUTION  The average is
\[ \frac{1}{1 - (-1)} \int_{-1}^{1} x^3 \, dx = \frac{1}{2} \int_{-1}^{1} x^3 \, dx = \frac{1}{8} x^4 \bigg|_{-1}^{1} = 0. \]

39. \( f(x) = \cos x, \quad [0, \frac{\pi}{2}] \)
SOLUTION  The average is
\[ \frac{1}{\pi/2 - 0} \int_{0}^{\pi/2} \cos x \, dx = \frac{2}{\pi} \left( \sin \frac{\pi}{2} \right) = \frac{2}{\pi}. \]

40. \( f(x) = \sec^2 x, \quad [0, \frac{\pi}{4}] \)
SOLUTION  The average is
\[ \frac{1}{\pi/4 - 0} \int_{0}^{\pi/4} \sec^2 x \, dx = \frac{4}{\pi} \tan x \bigg|_{0}^{\pi/4} = \frac{4}{\pi}. \]

41. \( f(s) = s^{-2}, \quad [2, 5] \)
SOLUTION  The average is
\[ \frac{1}{5 - 2} \int_{2}^{5} s^{-2} \, ds = -\frac{1}{3} s^{-1} \bigg|_{2}^{5} = \frac{1}{10}. \]

42. \( f(x) = \frac{\sin(\pi/x)}{x^2}, \quad [1, 2] \)
SOLUTION  The average is
\[ \frac{1}{2 - 1} \int_{1}^{2} \frac{\sin(\pi/x)}{x^2} \, dx = \frac{1}{\pi} \int_{\pi/2}^{\pi} \sin u \, du = -\frac{1}{\pi} \cos u \bigg|_{\pi/2}^{\pi} = \frac{1}{\pi}. \]

43. \( f(x) = 2x^3 - 3x^2, \quad [-1, 3] \)
SOLUTION  The average is
\[ \frac{1}{3 - (-1)} \int_{-1}^{3} (2x^3 - 3x^2) \, dx = \frac{1}{4} \left( \frac{1}{2} x^4 - x^3 \right) \bigg|_{-1}^{3} = 3. \]
44. Let \( f(x) = x^n \), \([0, 1]\)

**SOLUTION** For \( n > -1 \), the average is

\[
\frac{1}{1 - 0} \int_0^1 x^n \, dx = \int_0^1 x^n \, dx = \frac{1}{n + 1} x^{n+1} \bigg|_0^1 = \frac{1}{n + 1}.
\]

45. Let \( f(x) = \frac{1}{x^2 + 1} \), \([-1, 1]\)

**SOLUTION** The average is

\[
\frac{1}{1 - (-1)} \int_{-1}^1 \frac{1}{x^2 + 1} \, dx = \frac{1}{2} \left[ \tan^{-1} x \right]_{-1}^1 = \frac{1}{2} \left[ \left( \frac{\pi}{4} \right) - \left( \frac{\pi}{4} \right) \right] = \frac{\pi}{4}.
\]

46. Let \( f(x) = e^{-nx} \), \([-1, 1]\)

**SOLUTION** The average is

\[
\frac{1}{1 - (-1)} \int_{-1}^1 e^{-nx} \, dx = \frac{1}{2} \left( -\frac{1}{n} e^{-nx} \right) \bigg|_{-1}^1 = \frac{1}{2} \left( -\frac{1}{n} e^{-n} + \frac{1}{n} e^n \right) = \frac{1}{n} \sinh n.
\]

47. Let \( M \) be the average value of \( f(x) = x^3 \) on \([0, A]\), where \( A > 0 \). Which theorem guarantees that \( f(c) = M \) has a solution \( c \) in \([0, A]\)? Find \( c \).

**SOLUTION** The Mean Value Theorem for Integrals guarantees that \( f(c) = M \) has a solution \( c \) in \([0, A]\). With \( f(x) = x^3 \) on \([0, A]\),

\[
M = \frac{1}{A - 0} \int_0^A x^3 \, dx = \frac{1}{A} \left. \frac{1}{4} x^4 \right|_0^A = \frac{A^3}{4}.
\]

Solving \( f(c) = c^3 = \frac{A^3}{4} \) for \( c \) yields

\[
c = \frac{A}{\sqrt[3]{4}}.
\]

48. **CAS** Let \( f(x) = 2 \sin x - x \). Use a computer algebra system to plot \( f(x) \) and estimate:

(a) The positive root \( \alpha \) of \( f(x) \).
(b) The average value \( M \) of \( f(x) \) on \([0, \alpha]\).
(c) A value \( c \in [0, \alpha] \) such that \( f(c) = M \).

**SOLUTION** Let \( f(x) = 2 \sin x - x \). A graph of \( y = f(x) \) is shown below. From this graph, the positive root of \( f(x) \) appears to be roughly \( x = 1.9 \).

(a) Using a computer algebra system, solving the equation

\[
2 \sin \alpha - \alpha = 0
\]

yields \( \alpha = 1.895494267 \).

(b) The average value of \( f(x) \) on \([0, \alpha]\) is

\[
M = \frac{1}{\alpha - 0} \int_0^\alpha f(x) \, dx = 0.4439980667.
\]

(c) Solving

\[
f(c) = 2 \sin c - c = 0.4439980667
\]

yields either \( c = 0.4805683082 \) or \( c = 1.555776337 \).

49. Which of \( f(x) = x \sin^2 x \) and \( g(x) = x^2 \sin^2 x \) has a larger average value over \([0, 1]\)? Over \([1, 2]\)?
**SOLUTION** The functions $f$ and $g$ differ only in the power of $x$ multiplying $\sin^2 x$. It is also important to note that $\sin^2 x \geq 0$ for all $x$. Now, for each $x \in (0, 1)$, $x > x^2$ so

$$f(x) = x \sin^2 x > x^2 \sin^2 x = g(x).$$

Thus, over $[0, 1]$, $f(x)$ will have a larger average value than $g(x)$. On the other hand, for each $x \in (1, 2)$, $x^2 > x$, so

$$g(x) = x^2 \sin^2 x > x \sin^2 x = f(x).$$

Thus, over $[1, 2]$, $g(x)$ will have the larger average value.

50. Show that the average value of $f(x) = \frac{\sin x}{x}$ over $[\frac{\pi}{2}, \pi]$ is less than 0.41. Sketch the graph if necessary.

**SOLUTION** The figure below shows the graph of $f(x) = \frac{\sin x}{x}$ over the interval $[\frac{\pi}{2}, \pi]$—the solid curve—and the graph of the line through the points $(\frac{\pi}{2}, 2\pi)$ and $(\pi, 0)$—the dashed curve. Clearly, the area under the graph of $f(x)$ is smaller than the area under the graph of the line; that is,

$$\int_{\frac{\pi}{2}}^{\pi} f(x) \, dx \leq \frac{1}{2} \cdot \frac{\pi}{2} \cdot 2\pi = \frac{\pi}{2}.$$

Consequently,

$$\frac{1}{\pi - \frac{\pi}{2}} \int_{\frac{\pi}{2}}^{\pi} f(x) \, dx \leq \frac{1}{2} \cdot \frac{2}{\pi} = \frac{1}{\pi} \approx 0.318 < 0.41.$$

51. Sketch the graph of a function $f(x)$ such that $f(x) \geq 0$ on $[0, 1]$ and $f(x) \leq 0$ on $[1, 2]$, whose average on $[0, 2]$ is negative.

**SOLUTION** Many solutions will exist. One could be

52. Find the average of $f(x) = ax + b$ over the interval $[-M, M]$, where $a$, $b$, and $M$ are arbitrary constants.

**SOLUTION** The average is

$$\frac{1}{M - (-M)} \int_{-M}^{M} (ax + b) \, dx = \frac{1}{2M} \int_{-M}^{M} (ax + b) \, dx = \frac{1}{2M} \left[ \frac{a}{2} x^2 + bx \right]_{-M}^{M} = b.$$

53. The temperature $T(t)$ at time $t$ (in hours) in an art museum varies according to $T(t) = 70 + 5 \cos \left( \frac{\pi}{12} t \right)$. Find the average over the time periods $[0, 24]$ and $[2, 6]$.

**SOLUTION**

- The average temperature over the 24-hour period is

$$\frac{1}{24 - 0} \int_{0}^{24} \left( 70 + 5 \cos \left( \frac{\pi}{12} t \right) \right) \, dt = \frac{1}{24} \left[ 70t + \frac{60}{\pi} \sin \left( \frac{\pi}{12} t \right) \right]_{0}^{24} = 70^\circ F.$$
The average temperature over the 4-hour period is

\[
\frac{1}{6 - 2} \int_2^6 (70 + 5 \cos \left( \frac{\pi}{12} t \right)) \, dt = \frac{1}{4} \left( 70t + \frac{60}{\pi} \sin \left( \frac{\pi}{12} t \right) \right) \bigg|_2^6 = 72.4^\circ \text{F}. 
\]

54. A ball is thrown in the air vertically from ground level with initial velocity 64 ft/s. Find the average height of the ball over the time interval extending from the time of the ball’s release to its return to ground level. Recall that the height at time \( t \) is \( h(t) = 64t - 16t^2 \).

**SOLUTION** The ball is at ground level at time \( t = 0 \) and \( t = 4 \). The average height of the ball is

\[
\frac{1}{4 - 0} \int_0^4 h(t) \, dt = \frac{1}{4} \int_0^4 (64t - 16t^2) \, dt = \frac{1}{4} \left( 32t^2 - \frac{16}{3} t^3 \right) \bigg|_0^4 = \frac{128}{3} \text{ ft.} 
\]

55. What is the average area of the circle whose radius varies from 0 to 2?

**SOLUTION** The average area is

\[
\frac{1}{1 - 0} \int_0^1 \pi r^2 \, dr = \frac{\pi}{3} \bigg|_0^1 = \frac{\pi}{3}. 
\]

56. An object with zero initial velocity accelerates at a constant rate of 10 m/s². Find its average velocity during the first 15 s.

**SOLUTION** An acceleration \( a(t) = 10 \) gives \( v(t) = 10t + c \) for some constant \( c \) and zero initial velocity implies \( c = 0 \). Thus the average velocity is given by

\[
\frac{1}{15 - 0} \int_0^{15} 10t \, dt = \frac{1}{3} t^2 \bigg|_0^{15} = 75 \text{ m/s.} 
\]

57. The acceleration of a particle is \( a(t) = t - t^3 \) m/s² for \( 0 \leq t \leq 1 \). Compute the average acceleration and average velocity over the time interval \([0, 1]\), assuming that the particle’s initial velocity is zero.

**SOLUTION** The average acceleration is

\[
\frac{1}{1 - 0} \int_0^1 (t - t^3) \, dt = \left( \frac{1}{2} t^2 - \frac{1}{4} t^4 \right) \bigg|_0^1 = \frac{1}{4} \text{ m/s}^2. 
\]

An acceleration \( a(t) = t - t^3 \) with zero initial velocity gives

\[
v(t) = \frac{1}{2} t^2 - \frac{1}{4} t^4. 
\]

Thus the average velocity is given by

\[
\frac{1}{1 - 0} \int_0^1 \left( \frac{1}{2} t^2 - \frac{1}{4} t^4 \right) \, dt = \left( \frac{1}{6} t^3 - \frac{1}{20} t^5 \right) \bigg|_0^1 = \frac{7}{60} \text{ m/s.} 
\]

58. Let \( M \) be the average value of \( f(x) = x^4 \) on \([0, 3]\). Find a value of \( c \) in \([0, 3]\) such that \( f(c) = M \).

**SOLUTION** We have

\[
M = \frac{1}{3 - 0} \int_0^3 x^4 \, dx = \frac{1}{3} \int_0^3 x^4 \, dx = \frac{1}{15} x^5 \bigg|_0^3 = \frac{81}{5}. 
\]

Then \( M = f(c) = c^4 = \frac{81}{5} \) implies \( c = \frac{3}{\sqrt[4]{5}} = 2.006221 \).

59. Let \( f(x) = \sqrt{x} \). Find a value of \( c \) in \([4, 9]\) such that \( f(c) \) is equal to the average of \( f \) on \([4, 9]\).

**SOLUTION** The average value is

\[
\frac{1}{9 - 4} \int_4^9 \sqrt{x} \, dx = \frac{1}{5} \int_4^9 \sqrt{x} \, dx = \frac{2}{15} x^{3/2} \bigg|_4^9 = \frac{38}{15}. 
\]

Then \( f(c) = \sqrt{c} = \frac{38}{15} \) implies

\[
c = \left( \frac{38}{15} \right)^2 = \frac{1444}{225} \approx 6.417778. 
\]
60. Give an example of a function (necessarily discontinuous) that does not satisfy the conclusion of the MVT for Integrals.

**SOLUTION** There are an infinite number of discontinuous functions that do not satisfy the conclusion of the Mean Value Theorem for Integrals. Consider the function on $[-1, 1]$ such that for $x < 0$, $f(x) = -1$ and for $x \geq 0$, $f(x) = 1$. Clearly the average value is 0 but $f(c) \neq 0$ for all $c \in [-1, 1]$.

**Further Insights and Challenges**

61. An object is tossed in the air vertically from ground level with initial velocity $v_0$ ft/s at time $t = 0$. Find the average speed of the object over the time interval $[0, T]$, where $T$ is the time the object returns to earth.

**SOLUTION** The height is given by $h(t) = v_0 t - 16t^2$. The ball is at ground level at time $t = 0$ and $T = v_0/16$. The velocity is given by $v(t) = v_0 - 32t$ and thus the speed is given by $s(t) = |v_0 - 32t|$. The average speed is

$$\frac{1}{v_0/16 - 0} \int_{0}^{v_0/16} |v_0 - 32t| \, dt = \frac{16}{v_0} \int_{0}^{v_0/32} (v_0 - 32t) \, dt + \frac{16}{v_0} \int_{v_0/32}^{v_0/16} (32t - v_0) \, dt$$

$$= \frac{16}{v_0} \left[ (v_0 t - 16t^2) \right]_{0}^{v_0/32} + \frac{16}{v_0} \left( 16t^2 - v_0 t \right)_{v_0/32}^{v_0/16} = \frac{v_0}{2}.$$

62. Review the MVT stated in Section 4.3 (Theorem 1, p. 230) and show how it can be used, together with the Fundamental Theorem of Calculus, to prove the MVT for integrals.

**SOLUTION** The Mean Value Theorem essentially states that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

for some $c \in (a, b)$. Let $F$ be any antiderivative of $f$. Then

$$f(c) = F'(c) = \frac{F(b) - F(a)}{b - a} = \frac{1}{b - a} (F(b) - F(a)) = \frac{1}{b - a} \int_{a}^{b} f(x) \, dx.$$

### 6.3 Volumes of Revolution

**Preliminary Questions**

1. Which of the following is a solid of revolution?
   (a) Sphere  (b) Pyramid  (c) Cylinder  (d) Cube

**SOLUTION** The sphere and the cylinder have circular cross sections; hence, these are solids of revolution. The pyramid and cube do not have circular cross sections, so these are not solids of revolution.

2. True or false? When a solid is formed by rotating the region under a graph about the $x$-axis, the cross sections perpendicular to the $x$-axis are circular disks.

**SOLUTION** True. The cross sections will be disks with radius equal to the value of the function.

3. True or false? When a solid is formed by rotating the region between two graphs about the $x$-axis, the cross sections perpendicular to the $x$-axis are circular disks.

**SOLUTION** False. The cross sections may be washers.

4. Which of the following integrals expresses the volume of the solid obtained by rotating the area between $y = f(x)$ and $y = g(x)$ over $[a, b]$ around the $x$-axis [assume $f(x) \geq g(x) \geq 0$]?
   (a) $\pi \int_{a}^{b} (f(x) - g(x))^2 \, dx$
   (b) $\pi \int_{a}^{b} (f(x))^2 - (g(x))^2 \, dx$

**SOLUTION** The correct answer is (b). Cross sections of the solid will be washers with outer radius $f(x)$ and inner radius $g(x)$. The area of the washer is then $\pi f(x)^2 - \pi g(x)^2 = \pi (f(x)^2 - g(x)^2)$. 

6. The ball is at ground level at time $t = 0$, $T$ is the time the object returns to earth.
Exercises

In Exercises 1–4, (a) sketch the solid obtained by revolving the region under the graph of \( f(x) \) about the \( x \)-axis over the given interval, (b) describe the cross section perpendicular to the \( x \)-axis located at \( x \), and (c) calculate the volume of the solid.

1. \( f(x) = x + 1, \quad [0, 3] \)

SOLUTION

(a) A sketch of the solid of revolution is shown below:

(b) Each cross section is a disk with radius \( x + 1 \).

(c) The volume of the solid of revolution is

\[
\pi \int_0^3 (x + 1)^2 \, dx = \pi \int_0^3 (x^2 + 2x + 1) \, dx = \pi \left( \frac{1}{3} x^3 + x^2 + x \right) \bigg|_0^3 = 21\pi.
\]

2. \( f(x) = x^2, \quad [1, 3] \)

SOLUTION

(a) A sketch of the solid of revolution is shown below:

(b) Each cross section is a disk of radius \( x^2 \).

(c) The volume of the solid of revolution is

\[
\pi \int_1^3 (x^2)^2 \, dx = \pi \left( \frac{x^5}{5} \right) \bigg|_1^3 = \frac{242\pi}{5}.
\]

3. \( f(x) = \sqrt{x + 1}, \quad [1, 4] \)

SOLUTION

(a) A sketch of the solid of revolution is shown below:

(b) Each cross section is a disk with radius \( \sqrt{x + 1} \).

(c) The volume of the solid of revolution is

\[
\pi \int_1^4 (\sqrt{x + 1})^2 \, dx = \pi \int_1^4 (x + 1) \, dx = \pi \left( \frac{1}{2} x^2 + x \right) \bigg|_1^4 = \frac{21\pi}{2}.
\]

4. \( f(x) = x^{-1}, \quad [1, 2] \)

SOLUTION

(a) A sketch of the solid of revolution is shown below:
(b) Each cross section is a disk with radius \( x^{-1} \).
(c) The volume of the solid of revolution is
\[
\pi \int_1^2 (x^{-1})^2 \, dx = \pi \int_1^2 x^{-2} \, dx = \pi \left( \frac{1}{-x} \right)^2 \bigg|_1^2 = \frac{\pi}{2}.
\]

In Exercises 5–12, find the volume of the solid obtained by rotating the region under the graph of the function about the \( x \)-axis over the given interval.

5. \( f(x) = x^2 - 3x, \quad [0, 3] \)

**SOLUTION** The volume of the solid of revolution is
\[
\pi \int_0^3 (x^2 - 3x)^2 \, dx = \pi \int_0^3 (x^4 - 6x^3 + 9x^2) \, dx = \pi \left( \frac{1}{5}x^5 - \frac{3}{2}x^4 + 3x^3 \right)^3_0 = \frac{81\pi}{10}.
\]

6. \( f(x) = \frac{1}{x^2}, \quad [1, 4] \)

**SOLUTION** The volume of the solid of revolution is
\[
\pi \int_1^4 (x^{-2})^2 \, dx = \pi \int_1^4 x^{-4} \, dx = \pi \left( \frac{1}{3}x^{-3} \right)^4_1 = \frac{21\pi}{64}.
\]

7. \( f(x) = x^{5/3}, \quad [1, 8] \)

**SOLUTION** The volume of the solid of revolution is
\[
\pi \int_1^8 (x^{5/3})^2 \, dx = \pi \int_1^8 x^{10/3} \, dx = \pi \left( \frac{3\pi}{13} \right)^{13/3}_1 = \frac{3\pi}{13}(2^{13} - 1) = 24573\pi.
\]

8. \( f(x) = 4 - x^2, \quad [0, 2] \)

**SOLUTION** The volume of the solid of revolution is
\[
\pi \int_0^2 (4 - x^2)^2 \, dx = \pi \int_0^2 (16 - 8x^2 + x^4) \, dx = \pi \left( 16x - \frac{8}{3}x^3 + \frac{1}{5}x^5 \right)^2_0 = \frac{256\pi}{15}.
\]

9. \( f(x) = \frac{2}{x+1}, \quad [1, 3] \)

**SOLUTION** The volume of the solid of revolution is
\[
\pi \int_1^3 \left( \frac{2}{x+1} \right)^2 \, dx = 4\pi \int_1^3 (x+1)^{-2} \, dx = -4\pi (x+1)^{-1} \bigg|_1^3 = \pi.
\]

10. \( f(x) = \sqrt{x^4 + 1}, \quad [1, 3] \)

**SOLUTION** The volume of the solid of revolution is
\[
\pi \int_1^3 (\sqrt{x^4 + 1})^2 \, dx = \pi \int_1^3 (x^4 + 1) \, dx = \pi \left( \frac{1}{5}x^5 + x \right)^3_1 = \frac{252\pi}{5}.
\]

11. \( f(x) = e^x, \quad [0, 1] \)

**SOLUTION** The volume of the solid of revolution is
\[
\pi \int_0^1 (e^x)^2 \, dx = \frac{1}{2} \pi e^{2x} \bigg|_0^1 = \frac{1}{2} \pi (e^2 - 1).
\]
12. \( f(x) = \sqrt{\cos x \sin x}, \quad [0, \pi/2] \)

**SOLUTION**

The volume of the solid of revolution is

\[
\pi \int_{0}^{\pi/2} (\sqrt{\cos x \sin x})^2 \, dx = \pi \int_{0}^{\pi/2} (\cos x \sin x) \, dx = \frac{\pi}{2} \left[ \int_{0}^{\pi/2} \sin 2x \, dx \right] = \frac{\pi}{2} \left( \frac{\pi}{2} \right)
\]

In Exercises 13–18, (a) sketch the region enclosed by the curves, (b) describe the cross section perpendicular to the \(x\)-axis located at \(x\), and (c) find the volume of the solid obtained by rotating the region about the \(x\)-axis.

13. \( y = x^2 + 2, \quad y = 10 - x^2 \)

**SOLUTION**

(a) Setting \( x^2 + 2 = 10 - x^2 \) yields \(2x^2 = 8, \) or \(x^2 = 4\). The two curves therefore intersect at \(x = \pm 2\). The region enclosed by the two curves is shown in the figure below.

(b) When the region is rotated about the \(x\)-axis, each cross section is a washer with outer radius \( R = 10 - x^2 \) and inner radius \( r = x^2 \).

(c) The volume of the solid of revolution is

\[
\pi \int_{-2}^{2} \left( (10 - x^2)^2 - (x^2 + 2)^2 \right) \, dx = \pi \int_{-2}^{2} (96 - 24x^2) \, dx = \pi \left[ 96x - 8x^3 \right]_{-2}^{2} = 256\pi.
\]

14. \( y = x^2, \quad y = 2x + 3 \)

**SOLUTION**

(a) Setting \( x^2 = 2x + 3 \) yields

\[
0 = x^2 - 2x - 3 = (x - 3)(x + 1).
\]

The two curves therefore intersect at \(x = -1\) and \(x = 3\). The region enclosed by the two curves is shown in the figure below.

(b) When the region is rotated about the \(x\)-axis, each cross section is a washer with outer radius \( R = 2x + 3 \) and inner radius \( r = x^2 \).

(c) The volume of the solid of revolution is

\[
\pi \int_{-1}^{3} \left( (2x + 3)^2 - (x^2)^2 \right) \, dx = \pi \int_{-1}^{3} (2x^2 + 12x + 9 - x^4) \, dx = \pi \left[ \frac{4}{3}x^3 + 6x^2 + 9x - \frac{1}{5}x^5 \right]_{-1}^{3} = \frac{1088\pi}{15}.
\]

15. \( y = 16 - x, \quad y = 3x + 12, \quad x = -1 \)

**SOLUTION**

(a) Setting \(16 - x = 3x + 12\), we find that the two lines intersect at \(x = 1\). The region enclosed by the two curves is shown in the figure below.
16. \( y = \frac{1}{x}, \quad y = 5 - x \)

**SOLUTION**

(a) Setting \( \frac{1}{x} = 5 - x \) yields

\[
0 = x^2 - \frac{5}{2}x + 1 = (x - 2)
\left( x - \frac{1}{2} \right).
\]

The two curves therefore intersect at \( x = 2 \) and \( x = \frac{1}{2} \). The region enclosed by the two curves is shown in the figure below.

(b) When the region is rotated about the \( x \)-axis, each cross section is a washer with outer radius \( R = \frac{5}{2} - x \) and inner radius \( r = x^{-1} \).

(c) The volume of the solid of revolution is

\[
\pi \int_{1/2}^{1} \left( \left( \frac{5}{2} - x \right)^{2} - \left( \frac{1}{x} \right)^{2} \right) \, dx = \pi \int_{1/2}^{1} \left( \frac{25}{4} - 5x + x^2 - x^{-2} \right) \, dx = \pi \left( \left. \left( \frac{25}{4}x - \frac{5}{2}x^2 + \frac{1}{3}x^3 + x^{-1} \right) \right|_{1/2}^{1} \right) = \frac{9\pi}{8}.
\]

17. \( y = \sec x, \quad y = 0, \quad x = -\frac{\pi}{4}, \quad x = \frac{\pi}{4} \)

**SOLUTION**

(a) The region in question is shown in the figure below.

(b) When the region is rotated about the \( x \)-axis, each cross section is a circular disk with radius \( R = \sec x \).
(c) The volume of the solid of revolution is
\[
\pi \int_{\pi/4}^{\pi/4} (\sec x)^2 \, dx = \pi \left( \tan x \right) \bigg|_{\pi/4}^{\pi/4} = 2\pi.
\]

18. \( y = \sec x, \quad y = \csc x, \quad y = 0, \quad x = 0, \quad \text{and} \quad x = \frac{\pi}{2}. \)

**SOLUTION**

(a) The region in question is shown in the figure below.

(b) When the region is rotated about the \( x \)-axis, cross sections for \( x \in [0, \pi/4] \) are circular disks with radius \( R = \sec x \), whereas cross sections for \( x \in [\pi/4, \pi/2] \) are circular disks with radius \( R = \csc x \).

(c) The volume of the solid of revolution is
\[
\pi \int_0^{\pi/4} \sec^2 x \, dx + \pi \int_{\pi/4}^{\pi/2} \csc^2 x \, dx = \pi \left( \tan x \right) \bigg|_0^{\pi/4} + \pi \left( -\cot x \right) \bigg|_{\pi/4}^{\pi/2} = \pi (1) + \pi (1) = 2\pi.
\]

In Exercises 19–22, find the volume of the solid obtained by rotating the region enclosed by the graphs about the \( y \)-axis over the given interval.

19. \( x = \sqrt{y}, \quad x = 0; \quad 1 \leq y \leq 4 \)

**SOLUTION** When the region in question (shown in the figure below) is rotated about the \( y \)-axis, each cross section is a disk with radius \( \sqrt{y} \). The volume of the solid of revolution is
\[
\pi \int_1^4 \left( \sqrt{y} \right)^2 \, dy = \pi \left( \frac{y^2}{2} \right) \bigg|_1^4 = \frac{15\pi}{2}.
\]

20. \( x = \sqrt{\sin y}, \quad x = 0; \quad 0 \leq y \leq \pi \)

**SOLUTION** When the region in question (shown in the figure below) is rotated about the \( y \)-axis, each cross section is a disk with radius \( \sqrt{\sin y} \). The volume of the solid of revolution is
\[
\pi \int_0^\pi \left( \sqrt{\sin y} \right)^2 \, dy = \pi \left( -\cos y \right) \bigg|_0^\pi = 2\pi.
\]
21. $x = y^2$, $x = \sqrt{y}$; $0 \leq y \leq 1$

**SOLUTION** When the region in question (shown in the figure below) is rotated about the $y$-axis, each cross section is a washer with outer radius $R = \sqrt{y}$ and inner radius $r = y^2$. The volume of the solid of revolution is

$$
\pi \int_0^1 \left( (\sqrt{y})^2 - (y^2)^2 \right) dy = \pi \left( \frac{y^2}{2} - \frac{y^5}{5} \right) \bigg|_0^1 = \frac{3\pi}{10}.
$$

![Image of washer](image)

22. $x = 4 - y$, $x = 16 - y^2$; $-3 \leq y \leq 4$

**SOLUTION** Setting $4 - y = 16 - y^2$ yields

$$
0 = y^2 - y - 12 = (y - 4)(y + 3),
$$

so the two curves intersect at $y = -3$ and $y = 4$. When the region enclosed by the two curves (shown in the figure below) is rotated about the $y$-axis, each cross section is a washer with outer radius $R = 16 - y^2$ and inner radius $r = 4 - y$.

The volume of the solid of revolution is

$$
\pi \int_{-3}^4 \left( (16 - y^2)^2 - (4 - y)^2 \right) dy = \pi \int_{-3}^4 \left( 16y^4 - 33y^2 + 8y + 240 \right) dy

= \pi \left( \frac{1}{5}y^5 - 11y^3 + 4y^2 + 240y \right) \bigg|_{-3}^4 = \frac{4802\pi}{5}.
$$

![Image of washer](image)

In Exercises 23–28, find the volume of the solid obtained by rotating region $A$ in Figure 10 about the given axis.

![Figure 10](image)

23. $x$-axis

**SOLUTION** Rotating region $A$ about the $x$-axis produces a solid whose cross sections are washers with outer radius $R = 6$ and inner radius $r = x^2 + 2$. The volume of the solid of revolution is

$$
\pi \int_0^2 \left( (6)^2 - (x^2 + 2)^2 \right) dx = \pi \int_0^2 \left( 32 - 4x^2 - x^4 \right) dx = \pi \left( 32x - \frac{4}{3}x^3 - \frac{1}{5}x^5 \right) \bigg|_0^2 = \frac{704\pi}{15}.
$$

24. $y = -2$

**SOLUTION** Rotating region $A$ about $y = -2$ produces a solid whose cross sections are washers with outer radius $R = 6 - (-2) = 8$ and inner radius $r = x^2 + 2 - (-2) = x^2 + 4$. The volume of the solid of revolution is

$$
\pi \int_0^2 \left( (8)^2 - (x^2 + 4)^2 \right) dx = \pi \int_0^2 \left( 48 - 8x^2 - x^4 \right) dx = \pi \left( 48x - \frac{8}{3}x^3 - \frac{1}{5}x^5 \right) \bigg|_0^2 = \frac{1024\pi}{15}.
$$
25. \( y = 2 \)

SOLUTION  Rotating the region \( A \) about \( y = 2 \) produces a solid whose cross sections are washers with outer radius \( R = 6 - 2 = 4 \) and inner radius \( r = x^2 + 2 - 2 = x^2 \). The volume of the solid of revolution is

\[
\pi \int_0^2 \left( 4^2 - (x^2)^2 \right) \, dx = \pi \left( 16x - \frac{1}{5}x^5 \right) \bigg|_0^2 = \frac{128\pi}{5}.
\]

26. \( y \)-axis

SOLUTION  Rotating region \( A \) about the \( y \)-axis produces a solid whose cross sections are disks with radius \( R = \sqrt{y - 2} \). Note that here we need to integrate along the \( y \)-axis. The volume of the solid of revolution is

\[
\pi \int_2^6 (\sqrt{y - 2})^2 \, dy = \pi \int_2^6 (y - 2) \, dy = \pi \left( \frac{1}{2}y^2 - 2y \right) \bigg|_2^6 = 8\pi.
\]

27. \( x = -3 \)

SOLUTION  Rotating region \( A \) about \( x = -3 \) produces a solid whose cross sections are washers with outer radius \( R = \sqrt{y - 2} - (-3) = \sqrt{y + 1} + 3 \) and inner radius \( r = 0 - (-3) = 3 \). The volume of the solid of revolution is

\[
\pi \int_2^6 \left( 3 + \sqrt{y - 2} \right)^2 - (3)^2 \, dy = \pi \int_2^6 (6\sqrt{y - 2} + y - 2) \, dy = \pi \left( 4(y - 2)^{3/2} + \frac{1}{2}y^2 - 2y \right) \bigg|_2^6 = 40\pi.
\]

28. \( x = 2 \)

SOLUTION  Rotating region \( A \) about \( x = 2 \) produces a solid whose cross sections are washers with outer radius \( R = 2 - 0 = 2 \) and inner radius \( r = 2 - \sqrt{y - 2} \). The volume of the solid of revolution is

\[
\pi \int_2^6 \left( 2^2 - (2 - \sqrt{y - 2})^2 \right) \, dy = \pi \int_2^6 \left( 4\sqrt{y - 2} - y + 2 \right) \, dy = \pi \left( \frac{8}{3}(y - 2)^{3/2} - \frac{1}{2}y^2 + 2y \right) \bigg|_2^6 = \frac{40\pi}{3}.
\]

In Exercises 29–34, find the volume of the solid obtained by rotating region \( B \) in Figure 10 about the given axis.

29. \( x \)-axis

SOLUTION  Rotating region \( B \) about the \( x \)-axis produces a solid whose cross sections are disks with radius \( R = x^2 + 2 \). The volume of the solid of revolution is

\[
\pi \int_0^2 (x^2 + 2)^2 \, dx = \pi \int_0^2 (x^4 + 4x^2 + 4) \, dx = \pi \left( \frac{1}{5}x^5 + \frac{4}{3}x^3 + 4x \right) \bigg|_0^2 = \frac{376\pi}{15}.
\]

30. \( y = -2 \)

SOLUTION  Rotating region \( B \) about \( y = -2 \) produces a solid whose cross sections are washers with outer radius \( R = x^2 + 2 - (-2) = x^2 + 4 \) and inner radius \( r = 0 - (-2) = 2 \). The volume of the solid of revolution is

\[
\pi \int_0^2 \left( (x^2 + 4)^2 - (2)^2 \right) \, dx = \pi \int_0^2 (x^4 + 8x^2 + 12) \, dx = \pi \left( \frac{1}{5}x^5 + \frac{8}{3}x^3 + 12x \right) \bigg|_0^2 = \frac{776\pi}{15}.
\]

31. \( y = 6 \)

SOLUTION  Rotating region \( B \) about \( y = 6 \) produces a solid whose cross sections are washers with outer radius \( R = 6 - 0 = 6 \) and inner radius \( r = 6 - (x^2 + 2) = 4 - x^2 \). The volume of the solid of revolution is

\[
\pi \int_0^2 \left( 6^2 - (4 - x^2)^2 \right) \, dy = \pi \int_0^2 (20 + 8x^2 - x^4) \, dy = \pi \left( 20x + \frac{8}{3}x^3 - \frac{1}{5}x^5 \right) \bigg|_0^2 = \frac{824\pi}{15}.
\]

32. \( y \)-axis

SOLUTION  Rotating region \( B \) about the \( y \)-axis produces a solid with two different cross sections. For each \( y \in [0, 2] \), the cross section is a disk with radius \( R = 2 \); for each \( y \in [2, 6] \), the cross section is a washer with outer radius \( R = 2 \) and inner radius \( r = \sqrt{y - 2} \). The volume of the solid of revolution is

\[
\pi \int_0^2 \left( 2 \right)^2 \, dy + \pi \int_2^6 \left( (2)^2 - (\sqrt{y - 2})^2 \right) \, dy = \pi \int_0^2 4 \, dy + \pi \int_2^6 (6 - y) \, dy
\]

\[
= \pi \left( 4y \right) \bigg|_0^2 + \pi \left( 6y - \frac{1}{2}y^2 \right) \bigg|_2^6 = 16\pi.
\]
Alternately, we recognize that rotating both region $A$ and region $B$ about the $y$-axis produces a cylinder of radius $R = 2$ and height $h = 6$. The volume of this cylinder is $\pi (2)^2 \cdot 6 = 24\pi$. In Exercise 26, we found that the volume of the solid generated by rotating region $A$ about the $y$-axis to be $8\pi$. Therefore, the volume of the solid generated by rotating region $B$ about the $y$-axis is $24\pi - 8\pi = 16\pi$.

**Hint for Exercise 32:** Express the volume as a sum of two integrals along the $y$-axis, or use Exercise 26.

33. $x = 2$

**SOLUTION** Rotating region $B$ about $x = 2$ produces a solid with two different cross sections. For each $y \in [0, 2]$, the cross section is a disk with radius $R = 2$; for each $y \in [2, 6]$, the cross section is a disk with radius $R = 2 - \sqrt{y - 2}$. The volume of the solid of revolution is

$$
\pi \int_0^2 (2)^2 \, dy + \pi \int_2^6 (2 - \sqrt{y - 2})^2 \, dy = \pi \int_0^2 4 \, dy + \pi \int_2^6 (2 + y - 4\sqrt{y - 2}) \, dy
$$

$$
= \pi (4y) \bigg|_0^2 + \pi \left( 2y + \frac{1}{2}y^2 - \frac{8}{3} (y - 2)^{3/2} \right) \bigg|_2^6 = \frac{32\pi}{3}.
$$

34. $x = -3$

**SOLUTION** Rotating region $B$ about $x = -3$ produces a solid with two different cross sections. For each $y \in [0, 2]$, the cross section is a washer with outer radius $R = 2 - (-3) = 5$ and inner radius $r = 0 - (-3) = 3$; for each $y \in [2, 6]$, the cross section is a washer with outer radius $R = 2 - (-3) = 5$ and inner radius $r = \sqrt{y - 2} - (-3) = \sqrt{y - 2} + 3$. The volume of the solid of revolution is

$$
\pi \int_0^2 (5^2 - (3^2)) \, dy + \pi \int_2^6 \left( 5^2 - (\sqrt{y - 2} + 3)^2 \right) \, dy
$$

$$
= \pi \int_0^2 16 \, dy + \pi \int_2^6 (18 + y - 6\sqrt{y - 2}) \, dy
$$

$$
= \pi (16y) \bigg|_0^2 + \pi \left( 18y - \frac{1}{2}y^2 - 4(y - 2)^{3/2} \right) \bigg|_2^6 = 56\pi.
$$

In Exercises 35–48, find the volume of the solid obtained by rotating the region enclosed by the graphs about the given axis.

35. $y = x^2, \quad y = 12 - x, \quad x = 0, \quad$ about $y = -2$

**SOLUTION** Rotating the region enclosed by $y = x^2$, $y = 12 - x$ and the $y$-axis (shown in the figure below) about $y = -2$ produces a solid whose cross sections are washers with outer radius $R = 12 - x - (-2) = 14 - x$ and inner radius $r = x^2 - (-2) = x^2 + 2$. The volume of the solid of revolution is

$$
\pi \int_0^3 \left( (14 - x)^2 - (x^2 + 2)^2 \right) \, dx = \pi \int_0^3 (192 - 28x - 3x^2 - x^4) \, dx
$$

$$
= \pi \left( 192x - 14x^2 - \frac{1}{5}x^5 \right) \bigg|_0^3 = \frac{1872\pi}{5}.
$$

36. $y = x^2, \quad y = 12 - x, \quad x = 0, \quad$ about $y = 15$

**SOLUTION** Rotating the region enclosed by $y = x^2$, $y = 12 - x$ and the $y$-axis (see the figure in the previous exercise) about $y = 15$ produces a solid whose cross sections are washers with outer radius $R = 15 - x^2$ and inner radius $r = 15 - (12 - x) = 3 + x$. The volume of the solid of revolution is

$$
\pi \int_0^3 \left( (15 - x^2)^2 - (3 + x)^2 \right) \, dx = \pi \int_0^3 (216 - 6x - 31x^2 + x^4) \, dx
$$
37. \( y = 16 - x, \quad y = 3x + 12, \quad x = 0, \) about \( y \)-axis

**SOLUTION** Rotating the region enclosed by \( y = 16 - x, \quad y = 3x + 12 \) and the \( y \)-axis (shown in the figure below) about the \( y \)-axis produces a solid with two different cross sections. For each \( y \in [12, 15] \), the cross section is a disk with radius \( R = \frac{1}{9}(y - 12) \); for each \( y \in [15, 16] \), the cross section is a disk with radius \( R = 16 - y \). The volume of the solid of revolution is

\[
\pi \int_{12}^{15} \left( \frac{1}{3}(y - 12) \right)^2 dy + \pi \int_{15}^{16} (16 - y)^2 dy
\]

\[
= \pi \int_{12}^{15} \frac{1}{9} (y^2 - 24y + 144) dy + \pi \int_{15}^{16} (y^2 - 32y + 256) dy
\]

\[
= \frac{\pi}{9} \left( \frac{1}{3}y^3 - 12y^2 + 144y \right)_{12}^{15} + \pi \left( \frac{1}{3}y^3 - 16y^2 + 256y \right)_{15}^{16} = \frac{4}{3}\pi.
\]

38. \( y = 16 - x, \quad y = 3x + 12, \quad x = 0, \quad x = 2 \)

**SOLUTION** Rotating the region enclosed by \( y = 16 - x, \quad y = 3x + 12 \) and the \( y \)-axis (see the figure in the previous exercise) about \( x = 2 \) produces a solid with two different cross sections. For each \( y \in [12, 15] \), the cross section is a washer with outer radius \( R = 2 \) and inner radius \( r = 2 - \frac{1}{9}(y - 12) = 6 - \frac{1}{9}y \); for each \( y \in [15, 16] \), the cross section is a washer with outer radius \( R = 2 \) and inner radius \( r = 2 - (16 - y) = y - 14 \). The volume of the solid of revolution is

\[
\pi \int_{12}^{15} \left( 2^2 - \left( 6 - \frac{1}{3}y \right)^2 \right) dy + \pi \int_{15}^{16} \left( 2^2 - (y - 14)^2 \right) dy
\]

\[
= \pi \int_{12}^{15} \left( -\frac{1}{9}y^2 + 4y - 32 \right) dy + \pi \int_{15}^{16} (-y^2 + 28y - 192) dy
\]

\[
= \pi \left( -\frac{1}{27}y^3 + 2y^2 - 32y \right)_{12}^{15} + \pi \left( -\frac{1}{3}y^3 + 14y^2 - 192y \right)_{15}^{16} = \frac{20}{3}\pi.
\]

39. \( y = \frac{9}{x^2}, \quad y = 10 - x^2, \quad \) about \( x \)-axis

**SOLUTION** The region enclosed by the two curves is shown in the figure below. Note that the region consists of two pieces that are symmetric with respect to the \( y \)-axis. We may therefore compute the volume of the solid of revolution by considering one of the pieces and doubling the result. Rotating the portion of the region in the first quadrant about the \( x \)-axis produces a solid whose cross sections are washers with outer radius \( R = 10 - x^2 \) and inner radius \( r = 9x^{-2} \). The volume of the solid of revolution is

\[
2\pi \int_{1}^{3} \left( (10 - x^2)^2 - (9x^{-2})^2 \right) dx = 2\pi \int_{1}^{3} \left( 100 - 20x^2 + x^4 - 81x^{-4} \right) dx
\]

\[
= 2\pi \left( 100x - \frac{20}{3}x^3 + \frac{1}{5}x^5 - 27x^{-3} \right)_{1}^{3} = \frac{1472\pi}{15}.
\]
40. \( y = \frac{9}{x^2}, \quad y = 10 - x^2 \), about \( y = 12 \)

**SOLUTION**  The region enclosed by the two curves is shown in the figure in the previous exercise. Note that the region consists of two pieces that are symmetric with respect to the \( y \)-axis. We may therefore compute the volume of the solid of revolution by considering one of the pieces and doubling the result. Rotating the portion of the region in the first quadrant about \( y = 12 \) produces a solid whose cross sections are washers with outer radius \( R = 12 - 9x^{-2} \) and inner radius \( r = 12 - (10 - x^2) = 2 + x^2 \). The volume of the solid of revolution is

\[
2\pi \int_{1}^{3} \left( (12 - 9x^{-2})^2 - (x^2 + 2)^2 \right) dx = 2\pi \int_{1}^{3} \left( 140 - 4x^2 - x^4 - 216x^{-2} + 81x^{-4} \right) dx
\]

\[
= 2\pi \left( 140x - \frac{4}{3}x^3 - \frac{1}{5}x^5 + 216x^{-1} - 27x^{-3} \right) \bigg|_{1}^{3} = \frac{2368\pi}{15}.
\]

41. \( y = \frac{1}{x}, \quad y = \frac{5}{2} - x \), about \( y \)-axis

**SOLUTION**  Rotating the region enclosed by \( y = x^{-1} \) and \( y = \frac{5}{2} - x \) (shown in the figure below) about the \( y \)-axis produces a solid whose cross sections are washers with outer radius \( R = \frac{5}{2} - y \) and inner radius \( r = y^{-1} \). The volume of the solid of revolution is

\[
\pi \int_{1/2}^{2} \left( \left( \frac{5}{2} - y \right)^2 - (y^{-1})^2 \right) dy = \pi \int_{1/2}^{2} \left( \frac{25}{4} - 5y + y^2 - y^{-2} \right) dy
\]

\[
= \pi \left( \frac{25}{4}y - \frac{5}{2}y^2 + \frac{1}{3}y^3 + y^{-1} \right) \bigg|_{1/2}^{2} = \frac{9\pi}{8}.
\]

42. \( x = 2, \quad x = 3, \quad y = 16 - x^4, \quad y = 0 \), about \( y \)-axis

**SOLUTION**  Rotating the region enclosed by \( x = 2, x = 3, y = 16 - x^4 \) and the \( x \)-axis (shown in the figure below) about the \( y \)-axis produces a solid whose cross sections are washers with outer radius \( R = 3 \) and inner radius \( r = \sqrt[3]{16} - y \). The volume of the solid of revolution is

\[
\pi \int_{-65}^{0} \left( 9 - \sqrt{16 - y} \right) dy = \left( 9y + \frac{2}{3}(16 - y)^{3/2} \right) \bigg|_{-65}^{0} = \frac{425\pi}{3}.
\]

43. \( y = x^3, \quad y = x^{1/3} \), about \( y \)-axis

**SOLUTION**  Rotating the region enclosed by \( y = x^3 \) and \( y = x^{1/3} \) (shown in the figure below) about the \( y \)-axis produces a solid whose cross sections are washers with outer radius \( R = y^{1/3} \) and inner radius \( r = y^3 \). The volume of the solid of revolution is

\[
\pi \int_{-1}^{1} \left( (y^{1/3})^2 - (y^3)^2 \right) dy = \pi \int_{-1}^{1} (y^{2/3} - y^6) dy = \pi \left( \frac{3}{5}y^{5/3} - \frac{1}{7}y^7 \right) \bigg|_{-1}^{1} = \frac{32\pi}{35}.
\]
44. \( y = x^3, \ y = x^{1/3}, \ \text{about } x = -2 \)

**SOLUTION** Rotating the region enclosed by \( y = x^3 \) and \( y = x^{1/3} \) (see the figure in the previous exercise) about \( x = -2 \) produces a solid with two different cross sections. For each \( y \in [-1, 0] \), the cross section is a washer with outer radius \( R = y^3 - (-2) = y^3 + 2 \) and inner radius \( r = y^{1/3} - (-2) = y^{1/3} + 2 \); for each \( y \in [0, 1] \), the cross section is a washer with outer radius \( R = y^{1/3} + 2 \) and inner radius \( r = y^3 + 2 \). The volume of the solid of revolution is

\[
\pi \int_{-1}^{1} [2 + y^3]^2 - (2 + \sqrt[3]{7})^2 \, dy + \pi \int_{0}^{1} (2 + \sqrt[3]{y})^2 - (2 + y^3)^2 \, dy
\]

\[= \pi \int_{-1}^{0} [4y^3 + y^6 - 4y^{1/3} - y^{2/3}] \, dy + \pi \int_{0}^{1} (4y^{1/3} + y^{2/3} - 4y^3 - y^6) \, dy
\]

\[= \pi \left[ \left. (y^4 + \frac{1}{7}y^7 - 3y^{4/3} - \frac{3}{5}y^{5/3}) \right|_{-1}^{0} + \pi \left. (3y^{4/3} + \frac{3}{5}y^{5/3} - y^4 - \frac{1}{7}y^7) \right|_{0}^{1} \right]
\]

\[= \frac{54\pi}{35} + \frac{86\pi}{35} = 4\pi.
\]

45. \( y = e^{-x}, \ y = 1 - e^{-x}, \ x = 0, \ \text{about } y = 4 \)

**SOLUTION** Rotating the region enclosed by \( y = 1 - e^{-x}, \ y = e^{-x} \) and the \( y \)-axis (shown in the figure below) about the line \( y = 4 \) produces a solid whose cross sections are washers with outer radius \( R = 4 - (1 - e^{-x}) = 3 + e^{-x} \) and inner radius \( r = 4 - e^{-x} \). The volume of the solid of revolution is

\[
\pi \int_{0}^{\ln 2} (3 + e^{-x})^2 - (4 - e^{-x})^2 \, dx = \pi \int_{0}^{\ln 2} (14e^{-x} - 7) \, dx
\]

\[= \pi \left[ (-7 - 7\ln 2 + 14) = 7\pi(1 - \ln 2). \right]
\]

46. \( y = \cosh x, \ x = \pm 2, \ \text{about } x\text{-axis} \)

**SOLUTION** Rotating the region enclosed by \( y = \cosh x, \ x = \pm 2 \) and the \( x \)-axis (shown in the figure below) about the \( x \)-axis produces a solid whose cross sections are disks with radius \( R = \cosh x \). The volume of the solid of revolution is

\[
\pi \int_{-2}^{2} \cosh^2 x \, dx = \frac{1}{2} \pi \int_{-2}^{2} (1 + \cosh 2x) \, dx = \frac{1}{2} \pi \left[ x + \frac{1}{2} \sinh 2x \right]_{-2}^{2}
\]

\[= \frac{1}{2} \pi \left[ (2 + \frac{1}{2} \sinh 4) - (-2 + \frac{1}{2} \sinh(-4)) \right] = \frac{1}{2} \pi(4 + \sinh 4).
\]
47. \( y^2 = 4x, \quad y = x, \quad y = 0, \) about \( x \)-axis

**SOLUTION** Rotating the region enclosed by \( y^2 = 4x \) and \( y = x \) (shown in the figure below) about the \( x \)-axis produces a solid whose cross sections are washers with outer radius \( R = 2\sqrt{x} \) and inner radius \( r = x \). The volume of the solid of revolution is

\[
\pi \int_0^4 \left( (2\sqrt{x})^2 - x^2 \right) dx = \pi \left| \frac{4x^2}{3} \right|_0^4 = \frac{32\pi}{3}.
\]

48. \( y^2 = 4x, \quad y = x, \) about \( y = 8 \)

**SOLUTION** Rotating the region enclosed by \( y^2 = 4x \) and \( y = x \) (see the figure in the previous exercise) about \( y = 8 \) produces a solid whose cross sections are washers with outer radius \( R = 8 - x \) and inner radius \( r = 8 - 2\sqrt{x} \). The volume of the solid of revolution is

\[
\pi \int_0^4 \left( (8 - x)^2 - (8 - \sqrt{4x})^2 \right) dx = \pi \left| \frac{x^3}{3} - 10x^2 + \frac{64}{3}x^{3/2} \right|_0^4 = 32\pi.
\]

49. **Sketch** the hypocycloid \( x^{2/3} + y^{2/3} = 1 \) and find the volume of the solid obtained by revolving it about the \( x \)-axis.

**SOLUTION** A sketch of the hypocycloid is shown below.

For the hypocycloid, \( y = \pm \left( 1 - x^{2/3} \right)^{3/2} \). Rotating this region about the \( x \)-axis will produce a solid whose cross sections are disks with radius \( R = \left( 1 - x^{2/3} \right)^{3/2} \). Thus the volume of the solid of revolution will be

\[
\pi \int_{-1}^1 \left( 1 - x^{2/3} \right)^{3/2} dx = \pi \left| \frac{3}{5}x^{5/3} - x^{10/3} + \frac{9}{7}x^{7/3} \right|_{-1}^1 = \frac{32\pi}{105}.
\]

50. The solid generated by rotating the region between the branches of the hyperbola \( y^2 - x^2 = 1 \) about the \( x \)-axis is called a **hyperboloid** (Figure 11). Find the volume of the hyperboloid for \(-a \leq x \leq a\).
Volumes of Revolution

6.3

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SOLUTION Each cross section is a disk of radius \( R = \sqrt{1 + x^2} \), so the volume of the hyperboloid is

\[
\pi \int_{-a}^{a} \left( \sqrt{1 + x^2} \right)^2 \, dx = \pi \int_{-a}^{a} \left( 1 + x^2 \right) \, dx = \pi \left( x + \frac{1}{3} x^3 \right) \bigg|_{-a}^{a} = \pi \left( \frac{2a^3 + 6a}{3} \right)
\]

51. A “bead” is formed by removing a cylinder of radius \( r \) from the center of a sphere of radius \( R \) (Figure 12). Find the volume of the bead with \( r = 1 \) and \( R = 2 \).

SOLUTION The equation of the outer circle is \( x^2 + y^2 = 2^2 \), and the inner cylinder intersects the sphere when \( y = \pm \sqrt{R^2 - r^2} \). Each cross section of the bead is a washer with outer radius \( \sqrt{R^2 - y^2} \) and inner radius \( r \), so the volume is given by

\[
\pi \int_{-\sqrt{R^2 - r^2}}^{\sqrt{R^2 - r^2}} \left( \sqrt{R^2 - y^2}^2 - r^2 \right) \, dy = \pi \int_{-\sqrt{R^2 - r^2}}^{\sqrt{R^2 - r^2}} \left( R^2 - y^2 \right) \, dy = 4\pi \sqrt{3}.
\]

Further Insights and Challenges

52. Find the volume \( V \) of the bead (Figure 12) in terms of \( r \) and \( R \). Then show that \( V = \frac{\pi}{6} h^3 \), where \( h \) is the height of the bead. This formula has a surprising consequence: Since \( V \) can be expressed in terms of \( h \) alone, it follows that two beads of height 2 in., one formed from a sphere the size of an orange and the other the size of the earth would have the same volume! Can you explain intuitively how this is possible?

SOLUTION The equation for the outer circle of the bead is \( x^2 + y^2 = R^2 \), and the inner cylinder intersects the sphere when \( y = \pm \sqrt{R^2 - r^2} \). Each cross section of the bead is a washer with outer radius \( \sqrt{R^2 - y^2} \) and inner radius \( r \), so the volume is

\[
\pi \int_{-\sqrt{R^2 - r^2}}^{\sqrt{R^2 - r^2}} \left( \sqrt{R^2 - y^2}^2 - r^2 \right) \, dy = \pi \int_{-\sqrt{R^2 - r^2}}^{\sqrt{R^2 - r^2}} \left( R^2 - y^2 \right) \, dy = \pi \left( R^2 - r^2 \right) \left( -\frac{1}{3} y^3 \right) \bigg|_{-\sqrt{R^2 - r^2}}^{\sqrt{R^2 - r^2}} = \frac{4}{3} \left( R^2 - r^2 \right)^{3/2} \pi.
\]

Now, \( h = 2\sqrt{R^2 - r^2} = 2(R^2 - r^2)^{1/2} \), which gives \( h^3 = 8(R^2 - r^2)^{3/2} \) and finally \( (R^2 - r^2)^{3/2} = \frac{1}{8} h^3 \). Substituting into the expression for the volume gives \( V = \frac{\pi}{6} h^3 \). The beads may have the same volume but clearly the wall of the earth-sized bead must be extremely thin while the orange-sized bead would be thicker.

53. The solid generated by rotating the region inside the ellipse with equation \( \left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2 = 1 \) around the \( x \)-axis is called an ellipsoid. Show that the ellipsoid has volume \( \frac{4}{3} \pi ab^2 \). What is the volume if the ellipse is rotated around the \( y \)-axis?

SOLUTION
• Rotating the ellipse about the $x$-axis produces an ellipsoid whose cross sections are disks with radius $R = b\sqrt{1 - (x/a)^2}$. The volume of the ellipsoid is then
\[
\pi \int_{-a}^{a} \left( b\sqrt{1 - (x/a)^2} \right)^2 \, dx = b^2 \pi \int_{-a}^{a} \left( 1 - \frac{1}{a^2} x^2 \right) \, dx = b^2 \pi \left( x - \frac{1}{3a^2} x^3 \right) \bigg|_{-a}^{a} = \frac{4}{3} \pi ab^2.
\]

• Rotating the ellipse about the $y$-axis produces an ellipsoid whose cross sections are disks with radius $R = a\sqrt{1 - (y/b)^2}$. The volume of the ellipsoid is then
\[
\int_{-b}^{b} \left( a\sqrt{1 - (y/b)^2} \right)^2 \, dy = a^2 \pi \int_{-b}^{b} \left( 1 - \frac{1}{b^2} y^2 \right) \, dy = a^2 \pi \left( y - \frac{1}{3b^2} y^3 \right) \bigg|_{-b}^{b} = \frac{4}{3} \pi a^2 b.
\]

54. A doughnut-shaped solid is called a **torus** (Figure 13). Use the washer method to calculate the volume of the torus obtained by rotating the region inside the circle with equation $(x - a)^2 + y^2 = b^2$ around the $y$-axis (assume that $a > b$).

**Hint:** Evaluate the integral by interpreting it as the area of a circle.

---

55. The curve $y = f(x)$ in Figure 14, called a **tractrix**, has the following property: the tangent line at each point $(x, y)$ on the curve has slope
\[
\frac{dy}{dx} = -\frac{y}{\sqrt{1 - y^2}}.
\]

Let $R$ be the shaded region under the graph of $0 \leq x \leq a$ in Figure 14. Compute the volume $V$ of the solid obtained by revolving $R$ around the $x$-axis in terms of the constant $c = f(a)$.

**Hint:** Use the disk method and the substitution $u = f(x)$ to show that
\[
V = \pi \int_{c}^{1} u\sqrt{1 - u^2} \, du
\]
SOLUTION Let \( y = f(x) \) be the tractrix depicted in Figure 14. Rotating the region \( R \) about the \( x \)-axis produces a solid whose cross sections are disks with radius \( f(x) \). The volume of the resulting solid is then

\[
V = \pi \int_0^a [f(x)]^2 \, dx.
\]

Now, let \( u = f(x) \). Then

\[
du = f'(x) \, dx = \frac{-f(x)}{\sqrt{1 - [f(x)]^2}} \, dx = \frac{-u}{\sqrt{1 - u^2}} \, dx;
\]

hence,

\[
dx = -\frac{\sqrt{1 - u^2}}{u} \, du,
\]

and

\[
V = \pi \int_1^c u^2 \left( \frac{\sqrt{1 - u^2}}{u} \, du \right) = \pi \int_1^c u \sqrt{1 - u^2} \, du.
\]

Carrying out the integration, we find

\[
V = -\frac{\pi}{3} (1 - u^2)^{3/2} \bigg|_1^c = \frac{\pi}{3} (1 - c^2)^{3/2}.
\]

56. Verify the formula

\[
\int_{x_1}^{x_2} (x - x_1)(x - x_2) \, dx = \frac{1}{6} (x_1 - x_2)^3
\]

Then prove that the solid obtained by rotating the shaded region in Figure 15 about the \( x \)-axis has volume \( V = \frac{\pi}{6} BH^2 \), with \( B \) and \( H \) as in the figure. Hint: Let \( x_1 \) and \( x_2 \) be the roots of \( f(x) = ax + b - (mx + c)^2 \), where \( x_1 < x_2 \). Show that

\[
V = \pi \int_{x_1}^{x_2} f(x) \, dx
\]

and use Eq. (3).

**FIGURE 15** The line \( y = mx + c \) intersects the parabola \( y^2 = ax + b \) at two points above the \( x \)-axis.

SOLUTION First, we calculate

\[
\int_{x_1}^{x_2} (x - x_1)(x - x_2) \, dx = \left[ \frac{1}{3} x^3 - \frac{1}{2} (x_1 + x_2)x^2 + x_1 x_2 x \right]_{x_1}^{x_2} = \frac{1}{6} x_1^3 - \frac{1}{2} x_1^2 x_2 + \frac{1}{2} x_1 x_2^2 - \frac{1}{6} x_2^3
\]

\[
= \frac{1}{6} \left( x_1^3 - 3x_1^2 x_2 + 3x_1 x_2^2 - x_2^3 \right) = \frac{1}{6} (x_1 - x_2)^3.
\]

Now, consider the region enclosed by the parabola \( y^2 = ax + b \) and the line \( y = mx + c \), and let \( x_1 \) and \( x_2 \) denote the \( x \)-coordinates of the points of intersection between the two curves with \( x_1 < x_2 \). Rotating the region about the \( y \)-axis produces a solid whose cross sections are washers with outer radius \( R = \sqrt{ax + b} \) and inner radius \( r = mx + c \). The volume of the solid of revolution is then

\[
V = \pi \int_{x_1}^{x_2} (ax + b - (mx + c)^2) \, dx
\]
Because $x_1$ and $x_2$ are roots of the equation $ax + b - (mx + c)^2 = 0$ and $ax + b - (mx + c)^2$ is a quadratic polynomial in $x$ with leading coefficient $-m^2$, it follows that $ax + b - (mx + c)^2 = -m^2(x - x_1)(x - x_2)$. Therefore,

$$V = -\pi m^2 \int_{x_1}^{x_2} (x - x_1)(x - x_2) \, dx = \frac{\pi}{6} m^2 (x_2 - x_1)^3,$$

where we have used Equation (3). From the diagram, we see that

$$B = x_2 - x_1 \quad \text{and} \quad H = mB,$$

so

$$V = \frac{\pi}{6} m^2 B^3 = \frac{\pi}{6} B (mB)^2 = \frac{\pi}{6} BH^2.$$

57. Let $R$ be the region in the unit circle lying above the cut with the line $y = mx + b$ (Figure 16). Assume the points where the line intersects the circle lie above the $x$-axis. Use the method of Exercise 56 to show that the solid obtained by rotating $R$ about the $x$-axis has volume $V = \frac{\pi}{6} h d^2$, with $h$ and $d$ as in the figure.

![Figure 16](image)

**SOLUTION** Let $x_1$ and $x_2$ denote the $x$-coordinates of the points of intersection between the circle $x^2 + y^2 = 1$ and the line $y = mx + b$ with $x_1 < x_2$. Rotating the region enclosed by the two curves about the $x$-axis produces a solid whose cross sections are washers with outer radius $R = \sqrt{1 - x^2}$ and inner radius $r = mx + b$. The volume of the resulting solid is then

$$V = \pi \int_{x_1}^{x_2} \left( (1 - x^2) - (mx + b)^2 \right) \, dx$$

Because $x_1$ and $x_2$ are roots of the equation $(1 - x^2) - (mx + b)^2 = 0$ and $(1 - x^2) - (mx + b)^2$ is a quadratic polynomial in $x$ with leading coefficient $-(1 + m^2)$, it follows that $(1 - x^2) - (mx + b)^2 = -(1 + m^2)(x - x_1)(x - x_2)$. Therefore,

$$V = -\pi(1 + m^2) \int_{x_1}^{x_2} (x - x_1)(x - x_2) \, dx = \frac{\pi}{6} (1 + m^2)(x_2 - x_1)^3.$$  

From the diagram, we see that $h = x_2 - x_1$. Moreover, by the Pythagorean theorem, $d^2 = h^2 + (mh)^2 = (1 + m^2)h^2$. Thus,

$$V = \frac{\pi}{6} (1 + m^2)h^3 = \frac{\pi}{6} h \left( (1 + m^2)h^2 \right) = \frac{\pi}{6} h d^2.$$

### 6.4 The Method of Cylindrical Shells

**Preliminary Questions**

1. Consider the region $R$ under the graph of the constant function $f(x) = h$ over the interval $[0, r]$. What are the height and radius of the cylinder generated when $R$ is rotated about:

   (a) the $x$-axis  
   (b) the $y$-axis

**SOLUTION**

(a) When the region is rotated about the $x$-axis, each shell will have radius $h$ and height $r$.

(b) When the region is rotated about the $y$-axis, each shell will have radius $r$ and height $h$.

2. Let $V$ be the volume of a solid of revolution about the $y$-axis.

   (a) Does the Shell Method for computing $V$ lead to an integral with respect to $x$ or $y$?
   (b) Does the Disk or Washer Method for computing $V$ lead to an integral with respect to $x$ or $y$?

**SOLUTION**
(a) The Shell method requires slicing the solid parallel to the axis of rotation. In this case, that will mean slicing the solid in the vertical direction, so integration will be with respect to $x$.

(b) The Disk or Washer method requires slicing the solid perpendicular to the axis of rotation. In this case, that means slicing the solid in the horizontal direction, so integration will be with respect to $y$.

Exercises

In Exercises 1–10, sketch the solid obtained by rotating the region underneath the graph of the function over the given interval about the $y$-axis and find its volume.

1. $f(x) = x^3$, $[0, 1]$

   **SOLUTION** A sketch of the solid is shown below. Each shell has radius $x$ and height $x^3$, so the volume of the solid is
   \[
   2\pi \int_0^1 x \cdot x^3 \, dx = 2\pi \int_0^1 x^4 \, dx = 2\pi \left. \left( \frac{1}{5} x^5 \right) \right|_0^1 = \frac{2}{5} \pi.
   \]

2. $f(x) = \sqrt{x}$, $[0, 4]$

   **SOLUTION** A sketch of the solid is shown below. Each shell has radius $x$ and height $\sqrt{x}$, so the volume of the solid is
   \[
   2\pi \int_0^4 x \sqrt{x} \, dx = 2\pi \int_0^4 x^{3/2} \, dx = 2\pi \left. \left( \frac{2}{5} x^{5/2} \right) \right|_0^4 = \frac{128}{5} \pi.
   \]

3. $f(x) = 3x + 2$, $[2, 4]$

   **SOLUTION** A sketch of the solid is shown below. Each shell has radius $x$ and height $3x + 2$, so the volume of the solid is
   \[
   2\pi \int_2^4 x(3x + 2) \, dx = 2\pi \int_2^4 (3x^2 + 2x) \, dx = 2\pi \left( \frac{1}{3} x^3 + x^2 \right) \bigg|_2^4 = 136 \pi.
   \]

4. $f(x) = 1 + x^2$, $[1, 3]$

   **SOLUTION** A sketch of the solid is shown below. Each shell has radius $x$ and height $1 + x^2$, so the volume of the solid is
   \[
   2\pi \int_1^3 x(1 + x^2) \, dx = 2\pi \int_1^3 (x + x^3) \, dx = 2\pi \left( \frac{1}{2} x^2 + \frac{1}{4} x^4 \right) \bigg|_1^3 = 48 \pi.
   \]
5. \( f(x) = 4 - x^2, \quad [0, 2] \)

**SOLUTION**  A sketch of the solid is shown below. Each shell has radius \( x \) and height \( 4 - x^2 \), so the volume of the solid is

\[
2\pi \int_0^2 x(4-x^2) \, dx = 2\pi \int_0^2 (4x-x^3) \, dx = 2\pi \left( \frac{2x^2}{2} - \frac{x^4}{4} \right) \bigg|_0^2 = 8\pi.
\]

6. \( f(x) = \sqrt{x^2+9}, \quad [0, 3] \)

**SOLUTION**  A sketch of the solid is shown below. Each shell has radius \( x \) and height \( \sqrt{x^2+9} \), so the volume of the solid is

\[
2\pi \int_0^3 x\sqrt{x^2+9} \, dx.
\]

Let \( u = x^2 + 9 \). Then \( du = 2x \, dx \) and

\[
2\pi \int_0^3 x\sqrt{x^2+9} \, dx = \pi \int_9^{18} \sqrt{u} \, du = \pi \left( \frac{2}{3} u^{3/2} \right) \bigg|_9^{18} = 18\pi(2\sqrt{3} - 1).
\]

7. \( f(x) = \sin(x^2), \quad [0, \sqrt{\pi}] \)

**SOLUTION**  A sketch of the solid is shown below. Each shell has radius \( x \) and height \( \sin x^2 \), so the volume of the solid is

\[
2\pi \int_0^{\sqrt{\pi}} x\sin(x^2) \, dx.
\]

Let \( u = x^2 \). Then \( du = 2x \, dx \) and

\[
2\pi \int_0^{\sqrt{\pi}} x\sin(x^2) \, dx = \pi \int_0^{\pi} \sin u \, du = -\pi (\cos u) \bigg|_0^{\pi} = 2\pi.
\]

---

\[y\]

\[x\]

5. \( f(x) = 4 - x^2, \quad [0, 2] \)

SOLUTION   A sketch of the solid is shown below. Each shell has radius \( x \) and height \( 4 - x^2 \), so the volume of the solid is

\[
2\pi \int_0^2 x(4-x^2) \, dx = 2\pi \int_0^2 (4x-x^3) \, dx = 2\pi \left( \frac{2x^2}{2} - \frac{x^4}{4} \right) \bigg|_0^2 = 8\pi.
\]

6. \( f(x) = \sqrt{x^2+9}, \quad [0, 3] \)

SOLUTION   A sketch of the solid is shown below. Each shell has radius \( x \) and height \( \sqrt{x^2+9} \), so the volume of the solid is

\[
2\pi \int_0^3 x\sqrt{x^2+9} \, dx.
\]

Let \( u = x^2 + 9 \). Then \( du = 2x \, dx \) and

\[
2\pi \int_0^3 x\sqrt{x^2+9} \, dx = \pi \int_9^{18} \sqrt{u} \, du = \pi \left( \frac{2}{3} u^{3/2} \right) \bigg|_9^{18} = 18\pi(2\sqrt{3} - 1).
\]

7. \( f(x) = \sin(x^2), \quad [0, \sqrt{\pi}] \)

SOLUTION   A sketch of the solid is shown below. Each shell has radius \( x \) and height \( \sin x^2 \), so the volume of the solid is

\[
2\pi \int_0^{\sqrt{\pi}} x\sin(x^2) \, dx.
\]

Let \( u = x^2 \). Then \( du = 2x \, dx \) and

\[
2\pi \int_0^{\sqrt{\pi}} x\sin(x^2) \, dx = \pi \int_0^{\pi} \sin u \, du = -\pi (\cos u) \bigg|_0^{\pi} = 2\pi.
\]
8. \( f(x) = x^{-1}, \quad [1, 3] \)

**SOLUTION** A sketch of the solid is shown below. Each shell has radius \( x \) and height \( x^{-1} \), so the volume of the solid is

\[
2\pi \int_1^3 x(x^{-1}) \, dx = 2\pi \int_1^3 1 \, dx = 2\pi (x) \bigg|_1^3 = 4\pi.
\]

9. \( f(x) = x + 1 - 2x^2, \quad [0, 1] \)

**SOLUTION** A sketch of the solid is shown below. Each shell has radius \( x \) and height \( x + 1 - 2x^2 \), so the volume of the solid is

\[
2\pi \int_0^1 x(x + 1 - 2x^2) \, dx = 2\pi \int_0^1 (x^2 + x - 2x^3) \, dx = 2\pi \left( \frac{1}{3}x^3 + \frac{1}{2}x^2 - \frac{1}{2}x^4 \right) \bigg|_0^1 = \frac{2}{3}\pi.
\]

10. \( f(x) = \frac{x}{\sqrt{1 + x^3}} \), \( [1, 4] \)

**SOLUTION** A sketch of the solid is shown below. Each shell has radius \( x \) and height \( \frac{x}{\sqrt{1 + x^3}} \), so the volume of the solid is

\[
2\pi \int_1^4 x \left( \frac{x}{\sqrt{1 + x^3}} \right) \, dx = 2\pi \int_1^4 \frac{x^2}{\sqrt{1 + x^3}} \, dx.
\]

Let \( u = 1 + x^3 \). Then \( du = 3x^2 \, dx \) and

\[
2\pi \int_1^4 \frac{x^2}{\sqrt{1 + x^3}} \, dx = \frac{2}{3} \pi \int_1^6 u^{-1/2} \, du = \frac{2}{3} \pi \left( 2u^{1/2} \right) \bigg|_2^6 = \frac{4\pi}{3} \left( \sqrt{65} - \sqrt{2} \right).
\]

In Exercises 11–14, use the Shell Method to compute the volume of the solids obtained by rotating the region enclosed by the graphs of the functions about the y-axis.

11. \( y = x^2, \quad y = 8 - x^2, \quad x = 0 \)

**SOLUTION** The region enclosed by \( y = x^2, \ y = 8 - x^2 \) and the y-axis is shown below. When rotating this region about the y-axis, each shell has radius \( x \) and height \( 8 - x^2 - x^2 = 8 - 2x^2 \). The volume of the resulting solid is

\[
2\pi \int_0^2 x(8 - 2x^2) \, dx = 2\pi \int_0^2 (8x - 2x^3) \, dx = 2\pi \left( \frac{4x^2}{2} - \frac{1}{2}x^4 \right) \bigg|_0^2 = 16\pi.
\]
12. \( y = 8 - x^3, \quad y = 8 - 4x \)

**SOLUTION**  The region enclosed by \( y = 8 - x^3 \) and \( y = 8 - 4x \) is shown below. When rotating this region about the \( y \)-axis, each shell has radius \( x \) and height \((8 - x^3) - (8 - 4x) = 4x - x^3 \). The volume of the resulting solid is

\[
2\pi \int_0^2 (4x - x^3) \, dx = 2\pi \left[ (4x^2/2 - x^4) \right]_0^2 = 2\pi \left( 8/3 - 1/5 \right) = \frac{128\pi}{15}.
\]

13. \( y = \sqrt{x}, \quad y = x^2 \)

**SOLUTION**  The region enclosed by \( y = \sqrt{x} \) and \( y = x^2 \) is shown below. When rotating this region about the \( y \)-axis, each shell has radius \( x \) and height \( \sqrt{x} - x^2 \). The volume of the resulting solid is

\[
2\pi \int_0^1 (x(\sqrt{x} - x^2)) \, dx = 2\pi \int_0^1 (x^{3/2} - x^3) \, dx = 2\pi \left( \frac{2}{3}x^{5/2} - \frac{1}{4}x^4 \right) \bigg|^1_0 = \frac{3\pi}{10}.
\]

14. \( y = 1 - |x - 1|, \quad y = 0 \)

**SOLUTION**  The region enclosed by \( y = 1 - |x - 1| \) and the \( x \)-axis is shown below. When rotating this region about the \( y \)-axis, two different shells are generated. For each \( x \in [0, 1] \), the shell has radius \( x \) and height \( x \); for each \( x \in [1, 2] \), the shell has radius \( x \) and height \( 2 - x \). The volume of the resulting solid is

\[
2\pi \int_0^1 x(x) \, dx + 2\pi \int_1^2 x(2 - x) \, dx = 2\pi \int_0^1 (x^2) \, dx + 2\pi \int_1^2 (2x - x^2) \, dx
\]

\[
= 2\pi \left( \frac{1}{3}x^3 \right) \bigg|_0^1 + 2\pi \left( x^2 - \frac{1}{3}x^3 \right) \bigg|_1^2 = 2\pi.
\]
In Exercises 15–16, use the Shell Method to compute the volume of rotation of the region enclosed by the curves about the y-axis. Use a computer algebra system or graphing utility to find the points of intersection numerically.

15. \(y = \frac{1}{2}x^2, \quad y = \sin(x^2)\)

**SOLUTION**  The region enclosed by \(y = \frac{1}{2}x^2\) and \(y = \sin(x^2)\) is shown below. When rotating this region about the y-axis, each shell has radius \(x\) and height \(\sin(x^2) - \frac{1}{2}x^2\). Using a computer algebra system, we find that the \(x\)-coordinate of the point of intersection on the right is \(x = 1.376769504\). Thus, the volume of the resulting solid of revolution is

\[
2\pi \int_0^{1.376769504} x \left( \sin(x^2) - \frac{1}{2}x^2 \right) \, dx = 1.321975576.
\]

16. \(y = e^{-x^2/2}, \quad y = x, \quad x = 0\)

**SOLUTION**  The region enclosed by \(y = e^{-x^2/2}, \ y = x\) and the y-axis is shown below. When rotating this region about the y-axis, each shell has radius \(x\) and height \(e^{-x^2/2} - x\). Using a computer algebra system, we find that the \(x\)-coordinate of the point of intersection on the right is \(x = 0.7530891650\). Thus, volume of the resulting solid of revolution is

\[
2\pi \int_0^{0.7530891650} x(e^{-x^2/2} - x) \, dx = 0.6568505551.
\]

In Exercises 17–22, sketch the solid obtained by rotating the region underneath the graph of the function over the interval about the given axis and calculate its volume using the Shell Method.

17. \(f(x) = x^3, \quad [0, 1], \ x = 2\)

**SOLUTION**  A sketch of the solid is shown below. Each shell has radius \(2 - x\) and height \(x^3\), so the volume of the solid is

\[
2\pi \int_0^1 (2 - x) \left( x^3 \right) \, dx = 2\pi \int_0^1 (2x^3 - x^4) \, dx = 2\pi \left( \frac{x^4}{2} - \frac{x^5}{5} \right)_0^1 = \frac{3\pi}{5}.
\]

18. \(f(x) = x^3, \quad [0, 1], \ x = -2\)

**SOLUTION**  A sketch of the solid is shown below. Each shell has radius \(x - (-2) = x + 2\) and height \(x^3\), so the volume of the solid is

\[
2\pi \int_0^1 (2 + x) \left( x^3 \right) \, dx = 2\pi \int_0^1 (2x^3 + x^4) \, dx = 2\pi \left( \frac{x^4}{2} + \frac{x^5}{5} \right)_0^1 = \frac{7\pi}{5}.
\]
19. $f(x) = x^{-4}$, $[-3, -1]$, $x = 4$

**Solution** A sketch of the solid is shown below. Each shell has radius $4 - x$ and height $x^{-4}$, so the volume of the solid is

$$2\pi \int_{-3}^{-1} (4 - x) \left( x^{-4} \right) \, dx = 2\pi \int_{-3}^{-1} (4x^{-4} - x^{-3}) \, dx = 2\pi \left( \frac{1}{2} x^{-2} - \frac{4}{3} x^{-3} \right) \bigg|_{-3}^{-1} = \frac{280\pi}{81}.$$ 

20. $f(x) = \frac{1}{\sqrt{x^2 + 1}}$, $[0, 2]$, $x = 0$

**Solution** A sketch of the solid is shown below. Each shell has radius $x$ and height $\frac{1}{\sqrt{x^2 + 1}}$, so the volume of the solid is

$$2\pi \int_{0}^{2} x \left( \frac{1}{\sqrt{x^2 + 1}} \right) \, dx = 2\pi \left( \sqrt{x^2 + 1} \right) \bigg|_{0}^{2} = 2\pi(\sqrt{3} - 1).$$ 

21. $f(x) = a - bx$, $[0, a/b]$, $x = -1$, $a, b > 0$

**Solution** A sketch of the solid is shown below. Each shell has radius $x - (-1) = x + 1$ and height $a - bx$, so the volume of the solid is

$$2\pi \int_{0}^{a/b} (x + 1) (a - bx) \, dx = 2\pi \int_{0}^{a/b} (a + (a - b)x - bx^2) \, dx = 2\pi \left( ax + \frac{a-b}{2} x^2 - \frac{b}{3} x^3 \right) \bigg|_{0}^{a/b} = 2\pi \left( \frac{a^2}{b} + \frac{a^3(a-b)}{2b^2} - \frac{a^3}{3b^2} \right) = a^2(a + 3b) \frac{3b^2}{2b^2} \pi.$$ 

22. $f(x) = 1 - x^2$, $[-1, 1]$, $x = c$ (with $c > 1$)
**SOLUTION** A sketch of the solid is shown below. Each shell has radius \( c - x \) and height \( 1 - x^2 \), so the volume of the solid is

\[
2\pi \int_{-1}^{1} (c - x) (1 - x^2) \, dx = 2\pi \int_{-1}^{1} (x^3 - cx^2 - x + c) \, dx = 2\pi \left( \frac{1}{4}x^4 - \frac{c}{3}x^3 - \frac{1}{2}x^2 + cx \right) \bigg|_{-1}^{1} = \frac{8c\pi}{3}.
\]

In Exercises 23–28, use the Shell Method to calculate the volume of rotation about the x-axis for the region underneath the graph.

23. \( y = x, \quad 0 \leq x \leq 1 \)

**SOLUTION** When the region shown below is rotated about the x-axis, each shell has radius \( y \) and height \( 1 - y \). The volume of the resulting solid is

\[
2\pi \int_{0}^{1} y(1 - y) \, dy = 2\pi \int_{0}^{1} (y - y^2) \, dy = 2\pi \left( \frac{1}{2}y^2 - \frac{1}{3}y^3 \right) \bigg|_{0}^{1} = \frac{\pi}{3}.
\]

24. \( y = 4 - x^2, \quad 0 \leq x \leq 2 \)

**SOLUTION** When the region shown below is rotated about the x-axis, each shell has radius \( y \) and height \( \sqrt{4 - y} \). The volume of the resulting solid is

\[
2\pi \int_{0}^{4} y \sqrt{4 - y} \, dy.
\]

Let \( u = 4 - y \). Then \( du = -dy \), \( y = 4 - u \), and

\[
2\pi \int_{0}^{4} y \sqrt{4 - y} \, dy = -2\pi \int_{0}^{4} (4 - u) \sqrt{u} \, du = 2\pi \int_{0}^{4} \left( 4\sqrt{u} - u^{3/2} \right) \, du
\]

\[
= 2\pi \left( \frac{8}{3}u^{3/2} - \frac{2}{5}u^{5/2} \right) \bigg|_{0}^{4} = \frac{256\pi}{15}.
\]

25. \( y = x^{1/3} - 2, \quad 8 \leq x \leq 27 \)

**SOLUTION** When the region shown below is rotated about the x-axis, each shell has radius \( y \) and height \( 27 - (y + 2)^3 \). The volume of the resulting solid is

\[
2\pi \int_{0}^{1} y \cdot \left( 27 - (y + 2)^3 \right) \, dy = 2\pi \int_{0}^{1} \left( 19y - 12y^2 - 6y^3 - y^4 \right) \, dy
\]
26. \( y = x^{-1}, \quad 1 \leq x \leq 4 \). Sketch the region and express the volume as a sum of two integrals.

**SOLUTION** When the region shown below is rotated about the \( x \)-axis, two different shells are generated. For each \( y \in [0, \frac{1}{4}] \), the shell has radius \( y \) and height \( 4 - 1 = 3 \); for each \( y \in [\frac{1}{4}, 1] \), the shell has radius \( y \) and height \( \frac{1}{y} - 1 \). The volume of the resulting solid is

\[
2\pi \int_{0}^{1/4} 3y \, dy + 2\pi \int_{1/4}^{1} y(y^{-1} - 1) \, dy = 2\pi \int_{0}^{1/4} 3y \, dy + 2\pi \int_{1/4}^{1} (1 - y) \, dy
\]

\[
= 2\pi \left( \frac{3}{2}y^2 \right) \bigg|_{0}^{1/4} + 2\pi \left( y - \frac{y^2}{2} \right) \bigg|_{1/4}^{1} = \frac{3\pi}{4}.
\]

27. \( y = x^{-2}, \quad 2 \leq x \leq 4 \)

**SOLUTION** When the region shown below is rotated about the \( x \)-axis, two different shells are generated. For each \( y \in [0, \frac{1}{16}] \), the shell has radius \( y \) and height \( 4 - 2 = 2 \); for each \( y \in [\frac{1}{16}, \frac{1}{4}] \), the shell has radius \( y \) and height \( \frac{1}{\sqrt{y}} - 2 \). The volume of the resulting solid is

\[
2\pi \int_{0}^{1/16} 2y \, dy + 2\pi \int_{1/16}^{1/4} y(y^{-1/2} - 2) \, dy = 2\pi \int_{0}^{1/16} 2y \, dy + 2\pi \int_{1/16}^{1/4} (y^{1/2} - 2y) \, dy
\]

\[
= 2\pi \left( y^2 \right) \bigg|_{0}^{1/16} + 2\pi \left( 2y^{3/2} - y^2 \right) \bigg|_{1/16}^{1/4} = \frac{\pi}{128} + \frac{11\pi}{384} = \frac{7\pi}{192}.
\]

28. \( y = \sqrt{x}, \quad 1 \leq x \leq 4 \)

**SOLUTION** When the region shown below is rotated about the \( x \)-axis, two different shells are generated. For each \( y \in [0, 1] \), the shell has radius \( y \) and height \( 4 - 1 = 3 \); for each \( y \in [1, 2] \), the shell has radius \( y \) and height \( 4 - y^2 \). The volume of the resulting solid is

\[
2\pi \int_{0}^{1} 3y \, dy + 2\pi \int_{1}^{2} y \left( 4 - y^2 \right) \, dy = 2\pi \int_{0}^{1} 3y \, dy + 2\pi \int_{1}^{2} (4y - y^3) \, dy
\]
29. Use both the Shell and Disk Methods to calculate the volume of the solid obtained by rotating the region under the graph of \( f(x) = 8 - x^3 \) for \( 0 \leq x \leq 2 \) about:
(a) the \( x \)-axis
(b) the \( y \)-axis

**SOLUTION**

(a) \( x \)-axis: Using the disk method, the cross sections are disks with radius \( R = 8 - x^3 \); hence the volume of the solid is
\[
\pi \int_0^2 (8 - x^3)^2 \, dx = \pi \left( 64x - 4x^4 + \frac{1}{7}x^7 \right) \bigg|_0^2 = \frac{576\pi}{7}.
\]

With the shell method, each shell has radius \( y \) and height \((8 - y)^{1/3}\). The volume of the solid is
\[
2\pi \int_0^8 y (8 - y)^{1/3} \, dy
\]
Let \( u = 8 - y \). Then \( dy = -du \), and
\[
2\pi \int_0^8 (8 - y)^{1/3} \, dy = 2\pi \int_0^8 (8 - u)^{1/3} \, du = 2\pi \int_0^8 (8u^{1/3} - u^{4/3}) \, du
= 2\pi \left( \frac{6u^{4/3}}{7} - \frac{3}{7}u^{7/3} \right) \bigg|_0^8 = \frac{576\pi}{7}.
\]

(b) \( y \)-axis: With the shell method, each shell has radius \( x \) and height \( 8 - x^3 \). The volume of the solid is
\[
2\pi \int_0^2 x(8 - x^3) \, dx = 2\pi \left( 4x^2 - \frac{1}{5}x^5 \right) \bigg|_0^2 = \frac{96\pi}{5}.
\]

Using the disk method, the cross sections are disks with radius \( R = (8 - y)^{1/3} \). The volume is then given by
\[
\pi \int_0^8 (8 - y)^{2/3} \, dy = -\frac{3\pi}{5} (8 - y)^{5/3} \bigg|_0^8 = \frac{96\pi}{5}.
\]

30. Sketch the solid of rotation about the \( y \)-axis for the region under the graph of the constant function \( f(x) = c \) (where \( c > 0 \)) for \( 0 \leq x \leq r \).
(a) Find the volume without using integration.
(b) Use the Shell Method to compute the volume.

**SOLUTION**

(a) The solid is simply a cylinder with height \( c \) and radius \( r \). The volume is given by \( \pi r^2 c \).
(b) Each shell has radius \( x \) and height \( c \), so the volume is
\[
2\pi \int_0^r cx \, dx = 2\pi \left( \frac{1}{2} x^2 \right) \bigg|_0^r = \pi r^2 c.
\]
31. Assume that the graph in Figure 9(A) can be described by both \( y = f(x) \) and \( x = h(y) \). Let \( V \) be the volume of the solid obtained by rotating the region under the curve about the \( y \)-axis.
   (a) Describe the figures generated by rotating segments \( \overline{AB} \) and \( \overline{CB} \) about the \( y \)-axis.
   (b) Set up integrals that compute \( V \) by the Shell and Disk Methods.

   ![Figure 9](image)

### SOLUTION

(a) When rotated about the \( y \)-axis, the segment \( \overline{AB} \) generates a disk with radius \( R = h(y) \) and the segment \( \overline{CB} \) generates a shell with radius \( x \) and height \( f(x) \).

(b) Based on Figure 9(A) and the information from part (a), when using the Shell Method,

\[
V = 2\pi \int_0^2 xf(x) \, dx;
\]

when using the Disk Method,

\[
V = \pi \int_0^{1.3} (h(y))^2 \, dy.
\]

32. Let \( W \) be the volume of the solid obtained by rotating the region under the curve in Figure 9(B) about the \( y \)-axis.
   (a) Describe the figures generated by rotating segments \( \overline{A'B'} \) and \( \overline{A'C'} \) about the \( y \)-axis.
   (b) Set up an integral that computes \( W \) by the Shell Method.
   (c) Explain the difficulty in computing \( W \) by the Washer Method.

### SOLUTION

(a) When rotated about the \( y \)-axis, the segment \( \overline{A'B'} \) generates a washer and the segment \( \overline{C'A'} \) generates a shell with radius \( x \) and height \( g(x) \).

(b) Using Figure 9(B) and the information from part (a),

\[
W = 2\pi \int_0^2 xg(x) \, dx.
\]

(c) The function \( g(x) \) is not one-to-one, which makes it difficult to determine the inner and outer radius of each washer.

In Exercises 33–38, use the Shell Method to find the volume of the solid obtained by rotating region \( A \) in Figure 10 about the given axis.

33. \( y \)-axis
   
   **SOLUTION**  When rotating region \( A \) about the \( y \)-axis, each shell has radius \( x \) and height \( 6 - (x^2 + 2) = 4 - x^2 \). The volume of the resulting solid is

\[
2\pi \int_0^2 x(4 - x^2) \, dx = 2\pi \int_0^2 (4x - x^3) \, dx = 2\pi \left( 2x^2 - \frac{1}{4}x^4 \right) \bigg|_0^2 = 8\pi.
\]

34. \( x = -3 \)
SOLUTION When rotating region A about \( x = -3 \), each shell has radius \( x - (-3) = x + 3 \) and height \( 6 - (x^2 + 2) = 4 - x^2 \). The volume of the resulting solid is
\[
2\pi \int_0^2 (x + 3)(4 - x^2) \, dx = 2\pi \int_0^2 (4x - x^3 + 12 - 3x^2) \, dx = 2\pi \left( 2x^2 - \frac{1}{4} x^4 + 12x - x^3 \right) \bigg|_0^2 = 40\pi.
\]

35. \( x = 2 \)

SOLUTION When rotating region A about \( x = 2 \), each shell has radius \( 2 - x \) and height \( 6 - (x^2 + 2) = 4 - x^2 \). The volume of the resulting solid is
\[
2\pi \int_0^2 (2 - x) \left( 4 - x^2 \right) \, dx = 2\pi \int_0^2 (8 - 2x^2 - 4x + x^3) \, dx = 2\pi \left( 8x - \frac{2}{3} x^3 - 2x^2 + \frac{1}{4} x^4 \right) \bigg|_0^2 = \frac{40\pi}{3}.
\]

36. \( x \)-axis

SOLUTION When rotating region A about the \( x \)-axis, each shell has radius \( y \) and height \( \sqrt{y - 2} \). The volume of the resulting solid is
\[
2\pi \int_2^6 y \sqrt{y - 2} \, dy
\]

Let \( u = y - 2 \). Then \( du = dy \), \( y = u + 2 \) and
\[
2\pi \int_2^6 y \sqrt{y - 2} \, dy = 2\pi \int_0^4 (u + 2) \sqrt{u} \, du = 2\pi \left( \frac{2}{3} u^{5/2} + \frac{4}{5} u^{3/2} \right) \bigg|_0^4 = \frac{704\pi}{15}.
\]

37. \( y = -2 \)

SOLUTION When rotating region A about \( y = -2 \), each shell has radius \( y - (-2) = y + 2 \) and height \( \sqrt{y - 2} \). The volume of the resulting solid is
\[
2\pi \int_2^6 (y + 2) \sqrt{y - 2} \, dy
\]

Let \( u = y - 2 \). Then \( du = dy \), \( y = u + 2 \) and
\[
2\pi \int_2^6 (y + 2) \sqrt{y - 2} \, dy = 2\pi \int_0^4 (u + 4) \sqrt{u} \, du = 2\pi \left( \frac{2}{5} u^{5/2} + \frac{8}{3} u^{3/2} \right) \bigg|_0^4 = \frac{1024\pi}{15}.
\]

38. \( y = 6 \)

SOLUTION When rotating region A about \( y = 6 \), each shell has radius \( 6 - y \) and height \( \sqrt{y - 2} \). The volume of the resulting solid is
\[
2\pi \int_2^6 (6 - y) \sqrt{y - 2} \, dy
\]

Let \( u = y - 2 \). Then \( du = dy \), \( 6 - y = 4 - u \) and
\[
2\pi \int_2^6 (6 - y) \sqrt{y - 2} \, dy = 2\pi \int_0^4 (4 - u) \sqrt{u} \, du = 2\pi \left( \frac{8}{3} u^{3/2} - \frac{2}{5} u^{5/2} \right) \bigg|_0^4 = \frac{256\pi}{15}.
\]

In Exercises 39–44, use the Shell Method to find the volumes of the solids obtained by rotating region B in Figure 10 about the given axis.

39. \( y \)-axis

SOLUTION When rotating region B about the \( y \)-axis, each shell has radius \( x \) and height \( x^2 + 2 \). The volume of the resulting solid is
\[
2\pi \int_0^2 x(x^2 + 2) \, dx = 2\pi \int_0^2 (x^3 + 2x) \, dx = 2\pi \left( \frac{1}{4} x^4 + x^2 \right) \bigg|_0^2 = 16\pi.
\]

40. \( x = -3 \)
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SOLUTION When rotating region $B$ about $x = -3$, each shell has radius $x - (-3) = x + 3$ and height $x^2 + 2$. The volume of the resulting solid is

$$2\pi \int_0^2 (x + 3)(x^2 + 2) \, dx = 2\pi \int_0^2 (x^3 + 3x^2 + 2x + 6) \, dx = 2\pi \left( \frac{1}{4} x^4 + \frac{3}{2} x^3 + x^2 + 6x \right) \bigg|_0^2 = 56\pi.$$

41. $x = 2$

SOLUTION When rotating region $B$ about $x = 2$, each shell has radius $2 - x$ and height $x^2 + 2$. The volume of the resulting solid is

$$2\pi \int_0^2 (2 - x)(x^2 + 2) \, dx = 2\pi \int_0^2 (2x^2 - x^3 + 4 - 2x) \, dx = 2\pi \left( \frac{2}{3} x^3 - \frac{1}{4} x^4 + 4x - x^2 \right) \bigg|_0^2 = \frac{32\pi}{3}.$$  

42. $x$-axis

SOLUTION When rotating region $B$ about the $x$-axis, two different shells are generated. For each $y \in [0, 2]$, the resulting shell has radius $y$ and height 2; for each $y \in [2, 6]$, the resulting shell has radius $y$ and height $2 - \sqrt{y - 2}$. The volume of the solid is then

$$2\pi \int_0^2 2y \, dy + 2\pi \int_2^6 y(2 - \sqrt{y - 2}) \, dy = 2\pi \int_0^6 2y \, dy - 2\pi \int_2^6 y\sqrt{y - 2} \, dy = 72\pi - 2\pi \int_2^6 y\sqrt{y - 2} \, dy.$$

In the remaining integral, let $u = y - 2$, so $du = dy$ and $y = u + 2$. Then

$$2\pi \int_2^6 y\sqrt{y - 2} \, dy = 2\pi \int_0^4 (u + 2)\sqrt{u} \, du = 2\pi \left( \frac{2}{3} u^{3/2} + \frac{4}{3} u^{3/2} \right) \bigg|_0^4 = \frac{704\pi}{15}.$$

Finally, the volume of the solid is

$$72\pi - \frac{704\pi}{15} = \frac{376\pi}{15}.$$  

43. $y = -2$

SOLUTION When rotating region $B$ about $y = -2$, two different shells are generated. For each $y \in [0, 2]$, the resulting shell has radius $y - (-2) = y + 2$ and height 2; for each $y \in [2, 6]$, the resulting shell has radius $y - (-2) = y + 2$ and height $2 - \sqrt{y - 2}$. The volume of the solid is then

$$2\pi \int_0^2 2(y + 2) \, dy + 2\pi \int_2^6 (y + 2)(2 - \sqrt{y - 2}) \, dy = 2\pi \int_0^6 2(y + 2) \, dy - 2\pi \int_2^6 (y + 2)\sqrt{y - 2} \, dy$$

$$= 120\pi - 2\pi \int_2^6 (y + 2)\sqrt{y - 2} \, dy.$$  

In the remaining integral, let $u = y - 2$, so $du = dy$ and $y + 2 = u + 4$. Then

$$2\pi \int_2^6 (u + 2)\sqrt{u} \, du = 2\pi \int_0^4 (u + 4)\sqrt{u} \, du = 2\pi \left( \frac{2}{3} u^{3/2} + \frac{8}{3} u^{3/2} \right) \bigg|_0^4 = \frac{1024\pi}{15}.$$

Finally, the volume of the solid is

$$120\pi - \frac{1024\pi}{15} = \frac{776\pi}{15}.$$  

44. $y = 8$

SOLUTION When rotating region $B$ about $y = 8$, two different shells are generated. For each $y \in [0, 2]$, the resulting shell has radius $8 - y$ and height 2; for each $y \in [2, 6]$, the resulting shell has radius $8 - y$ and height $2 - \sqrt{y - 2}$. The volume of the solid is then

$$2\pi \int_0^2 2(8 - y) \, dy + 2\pi \int_2^6 (8 - y)(2 - \sqrt{y - 2}) \, dy = 2\pi \int_0^6 2(8 - y) \, dy - 2\pi \int_2^6 (8 - y)\sqrt{y - 2} \, dy$$

$$= 120\pi - 2\pi \int_2^6 (8 - y)\sqrt{y - 2} \, dy.$$  

In the remaining integral, let $u = y - 2$, so $du = dy$ and $8 - y = 6 - u$. Then

$$2\pi \int_2^6 (8 - y)\sqrt{y - 2} \, dy = 2\pi \int_0^4 (6 - u)\sqrt{u} \, du = 2\pi \left( 4u^{3/2} - \frac{2}{5} u^{5/2} \right) \bigg|_0^4 = \frac{192\pi}{5}.$$
Finally, the volume of the solid is

\[ 120\pi - \frac{192\pi}{5} = \frac{408\pi}{5}. \]

45. Use the Shell Method to compute the volume of a sphere of radius \( r \).

**SOLUTION** A sphere of radius \( r \) can be generated by rotating the region under the semicircle \( y = \sqrt{r^2 - x^2} \) around the \( x \)-axis. Each shell has radius \( y \) and height

\[ \sqrt{r^2 - y^2} - \left(-\sqrt{r^2 - y^2}\right) = 2\sqrt{r^2 - y^2}. \]

Thus, the volume of the sphere is

\[ 2\pi \int_0^r 2y\sqrt{r^2 - y^2} \, dy. \]

Let \( u = r^2 - y^2 \). Then \( du = -2y \, dy \) and

\[ 2\pi \int_0^r 2y\sqrt{r^2 - y^2} \, dy = 2\pi \int_0^r \sqrt{u} \, du = 2\pi \left( \frac{2}{3} u^{3/2} \right)_0^r = \frac{4}{3} \pi r^3. \]

46. Use the Shell Method to calculate the volume \( V \) of the “bead” formed by removing a cylinder of radius \( r \) from the center of a sphere of radius \( R \) (compare with Exercise 51 in Section 6.3).

**SOLUTION** Each shell has radius \( x \) and height \( 2\sqrt{R^2 - x^2} \). The volume of the bead is then

\[ 2\pi \int_r^R 2x\sqrt{R^2 - x^2} \, dx. \]

Let \( u = R^2 - x^2 \). Then \( du = -2x \, dx \) and

\[ 2\pi \int_r^R 2x\sqrt{R^2 - x^2} \, dx = 2\pi \int_0^{R^2-r^2} \sqrt{u} \, du = 2\pi \left( \frac{2}{3} u^{3/2} \right)_0^{R^2-r^2} = \frac{4}{3} \pi (R^2 - r^2)^{3/2}. \]

47. Use the Shell Method to compute the volume of the torus obtained by rotating the interior of the circle \( (x - a)^2 + y^2 = r^2 \) about the \( y \)-axis, where \( a > r \). Hint: Evaluate the integral by interpreting part of it as the area of a circle.

**SOLUTION** When rotating the region enclosed by the circle \( (x - a)^2 + y^2 = r^2 \) about the \( y \)-axis each shell has radius \( x \) and height

\[ \sqrt{r^2 - (x - a)^2} - \left(-\sqrt{r^2 - (x - a)^2}\right) = 2\sqrt{r^2 - (x - a)^2}. \]

The volume of the resulting torus is then

\[ 2\pi \int_{a-r}^{a+r} 2x\sqrt{r^2 - (x - a)^2} \, dx. \]

Let \( u = x - a \). Then \( du = dx \), \( x = u + a \) and

\[ 2\pi \int_{a-r}^{a+r} 2x\sqrt{r^2 - (x - a)^2} \, dx = 2\pi \int_{-r}^r 2(u + a)\sqrt{r^2 - u^2} \, du \]

\[ = 4\pi \int_{-r}^r u\sqrt{r^2 - u^2} \, du + 4a\pi \int_{-r}^r \sqrt{r^2 - u^2} \, du. \]

Now,

\[ \int_{-r}^r u\sqrt{r^2 - u^2} \, du = 0 \]

because the integrand is an odd function and the integration interval is symmetric with respect to zero. Moreover, the other integral is one-half the area of a circle of radius \( r \); thus,

\[ \int_{-r}^r \sqrt{r^2 - u^2} \, du = \frac{1}{2} \pi r^2. \]

Finally, the volume of the torus is

\[ 4\pi(0) + 4a\pi \left( \frac{1}{2} \pi r^2 \right) = 2\pi^2 ar^2. \]
48. Use the Shell or Disk Method (whichever is easier) to compute the volume of the solid obtained by rotating the region in Figure 11 about:

(a) the $x$-axis

(b) the $y$-axis

**SOLUTION** Examine Figure 11. If the indicated region is sliced vertically, then the top of the slice lies along the curve $y = x - x^{12}$ and the bottom lies along the curve $y = 0$ (the $x$-axis). On the other hand, if the region is sliced horizontally, the equation $y = x - x^{12}$ must be solved for $x$ in order to determine the endpoint locations. Clearly, it will be easier to slice the region vertically.

(a) Now, suppose the region in Figure 11 is rotated about the $x$-axis. Because a vertical slice is perpendicular to the $x$-axis, we will calculate the volume of the resulting solid using the disk method. Each cross section is a disk of radius $x = \frac{1}{3}x^3 - \frac{1}{7}x^{14} + \frac{1}{25}x^{25}$, so the volume is

$$\pi \int_{0}^{1} \left( x - x^{12} \right)^2 \, dx = \pi \left[ \frac{1}{3}x^3 - \frac{1}{7}x^{14} + \frac{1}{25}x^{25} \right]_{0}^{1} = \frac{121\pi}{525}.$$ 

(b) Now suppose the region is rotated about the $y$-axis. Because a vertical slice is parallel to the $y$-axis, we will calculate the volume of the resulting solid using the shell method. Each shell has radius $x$ and height $x - x^{12}$, so the volume is

$$2\pi \int_{0}^{1} x(x - x^{12}) \, dx = 2\pi \left[ \frac{1}{3}x^3 - \frac{1}{14}x^{14} \right]_{0}^{1} = \frac{11\pi}{21}.$$ 

49. Use the most convenient method to compute the volume of the solid obtained by rotating the region in Figure 12 about the axis:

(a) $x = 4$

(b) $y = -2$

**SOLUTION** Examine Figure 12. If the indicated region is sliced vertically, then the top of the slice lies along the curve $y = x^3 + 2$ and the bottom lies along the curve $y = 4 - x^2$. On the other hand, the left end of a horizontal slice switches from $y = 4 - x^2$ to $y = x^3 + 2$ at $y = 3$. Here, vertical slices will be more convenient.

(a) Now, suppose the region in Figure 12 is rotated about $x = 4$. Because a vertical slice is parallel to $x = 4$, we will calculate the volume of the resulting solid using the shell method. Each shell has radius $4 - x$ and height $x^3 + 2 - (4 - x^2) = x^3 + x^2 - 2$, so the volume is

$$2\pi \int_{1}^{2} (4 - x)(x^3 + x^2 - 2) \, dx = 2\pi \left[ -\frac{1}{5}x^5 + \frac{3}{4}x^4 + \frac{4}{3}x^3 + x^2 - 8x \right]_{1}^{2} = \frac{563\pi}{30}.$$ 

(b) Now suppose the region is rotated about $y = -2$. Because a vertical slice is perpendicular to $y = -2$, we will calculate the volume of the resulting solid using the disk method. Each cross section is a washer with outer radius $R = x^3 + 2 - (-2) = x^3 + 4$ and inner radius $r = 4 - x^2 - (-2) = 6 - x^2$, so the volume is

$$\pi \int_{1}^{2} \left( (x^3 + 4)^2 - (6 - x^2)^2 \right) \, dx = \pi \left[ \frac{1}{7}x^7 - \frac{1}{5}x^5 + 2x^4 + 4x^3 - 20x \right]_{1}^{2} = \frac{1748\pi}{35}.$$
**Further Insights and Challenges**

50. The surface area of a sphere of radius \( r \) is \( 4\pi r^2 \). Use this to derive the formula for the volume \( V \) of a sphere of radius \( R \) in a new way.

(a) Show that the volume of a thin spherical shell of inner radius \( r \) and thickness \( \Delta x \) is approximately \( 4\pi r^2 \Delta x \).

(b) Approximate \( V \) by decomposing the sphere of radius \( R \) into \( N \) thin spherical shells of thickness \( \Delta x = R/N \).

(c) Show that the approximation is a Riemann sum which converges to an integral. Evaluate the integral.

**SOLUTION**

(a) The volume of a thin spherical shell of inner radius \( r \) and thickness \( \Delta x \) is given by the product of the surface area of the shell, \( 4\pi r^2 \), and the thickness. Thus, we have \( 4\pi r^2 \Delta x \).

(b) The volume of the sphere is approximated by

\[
V \approx 4\pi \frac{R}{N} \sum_{k=1}^{N} (x_k)^2
\]

where \( x_k = k \frac{R}{N} \).

(c) \( V = 4\pi \lim_{N \to \infty} \left( \frac{R}{N} \right) \sum_{k=1}^{N} (x_k)^2 = 4\pi \int_{0}^{R} x^2 \, dx = 4\pi \left( \frac{1}{3} R^3 \right) \bigg|_{0}^{R} = \frac{4}{3} \pi R^3 \).

51. Let \( R \) be the region bounded by the ellipse \( \left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2 = 1 \) (Figure 13). Show that the solid obtained by rotating \( R \) about the \( y \)-axis (called an ellipsoid) has volume \( \frac{4}{3} \pi a^2 b \).

**SOLUTION** Let’s slice the portion of the ellipse in the first and fourth quadrants horizontally and rotate the slices about the \( y \)-axis. The resulting ellipsoid has cross sections that are disks with radius

\[
R = \sqrt{a^2 - \frac{a^2 y^2}{b^2}}.
\]

Thus, the volume of the ellipsoid is

\[
\pi \int_{-b}^{b} \left( a^2 - \frac{a^2 y^2}{b^2} \right) \, dy = \pi \left( a^2 y - \frac{a^2 y^3}{3 b^2} \right) \bigg|_{-b}^{b} = \pi \left[ \left( a^2 b - \frac{a^2 b}{3} \right) - \left( -a^2 b + \frac{a^2 b}{3} \right) \right] = \frac{4}{3} \pi a^2 b.
\]

52. The bell-shaped curve in Figure 14 is the graph of a certain function \( y = f(x) \) with the following property: The tangent line at a point \( (x, y) \) on the graph has slope \( dy/dx = -xy \). Let \( R \) be the shaded region under the graph for \( 0 \leq x \leq a \) in Figure 14. Use the Shell Method and the substitution \( u = f(x) \) to show that the solid obtained by revolving \( R \) around the \( y \)-axis has volume \( V = 2\pi (1 - c) \), where \( c = f(a) \). Observe that as \( c \to 0 \), the region \( R \) becomes infinite but the volume \( V \) approaches \( 2\pi \).

**FIGURE 14** The bell-shaped curve.
Let \( y = f(x) \) be the exponential function depicted in Figure 14. When rotating the region \( R \) about the \( y \)-axis, each shell in the resulting solid has radius \( x \) and height \( f(x) \). The volume of the solid is then

\[
V = 2\pi \int_0^a xf(x) \, dx.
\]

Now, let \( u = f(x) \). Then \( du = f'(x) \, dx = -xf(x) \, dx \); hence, \( xf(x) \, dx = -du \), and

\[
V = 2\pi \int_C (-du) = 2\pi \int_1^c du = 2\pi(1 - c).
\]

### 6.5 Work and Energy

**Preliminary Questions**

1. Why is integration needed to compute the work performed in stretching a spring?

**SOLUTION** Recall that the force needed to extend or compress a spring depends on the amount by which the spring has already been extended or compressed from its equilibrium position. In other words, the force needed to move a spring is variable. Whenever the force is variable, work needs to be computed with an integral.

2. Why is integration needed to compute the work performed in pumping water out of a tank but not to compute the work performed in lifting up the tank?

**SOLUTION** To lift a tank through a vertical distance \( d \), the force needed to move the tank remains constant; hence, no integral is needed to calculate the work done in lifting the tank. On the other hand, pumping water from a tank requires that different layers of the water be moved through different distances, and, depending on the shape of the tank, may require different forces. Thus, pumping water from a tank requires that an integral be evaluated.

3. Which of the following represents the work required to stretch a spring (with spring constant \( k \)) a distance \( x \) beyond its equilibrium position: \( kx \), \(-kx\), \( \frac{1}{2}mk^2 \), \( \frac{1}{2}kx^2 \), or \( \frac{1}{2}mx^2 \)?

**SOLUTION** The work required to stretch a spring with spring constant \( k \) a distance \( x \) beyond its equilibrium position is

\[
\int_0^x ky \, dy = \frac{1}{2}ky^2 \Big|_0^x = \frac{1}{2}kx^2.
\]

**Exercises**

1. How much work is done raising a 4-kg mass to a height of 16 m above ground?

**SOLUTION** The force needed to lift a 4-kg object is a constant

\[(4 \text{ kg})(9.8 \text{ m/s}^2) = 39.2 \text{ N}.
\]

The work done in lifting the object to a height of 16 m is then

\[(39.2 \text{ N})(16 \text{ m}) = 627.2 \text{ J}.
\]

2. How much work is done raising a 4-lb mass to a height of 16 ft above ground?

**SOLUTION** The force needed to lift a 4-lb object is a constant 4 lb. The work done in lifting the object to a height of 16 ft is then

\[(4 \text{ lb})(16 \text{ ft}) = 64 \text{ ft-lb}.
\]

**In Exercises 3–6, compute the work (in joules) required to stretch or compress a spring as indicated, assuming that the spring constant is \( k = 150 \text{ kg/s}^2 \).**

3. Stretching from equilibrium to 12 cm past equilibrium

**SOLUTION** The work required to stretch the spring 12 cm past equilibrium is

\[
\int_0^{0.12} 150x \, dx = 75x^2 \Big|_0^{0.12} = 1.08 \text{ J}.
\]

4. Compressing from equilibrium to 4 cm past equilibrium
SOLUTION The work required to compress the spring 4 cm past equilibrium is

\[
\int_{0}^{-0.04} 150x \, dx = 75x^2 \bigg|_{0}^{-0.04} = 0.12 \text{ J}.
\]

5. Stretching from 5 to 15 cm past equilibrium

SOLUTION The work required to stretch the spring from 5 cm to 15 cm past equilibrium is

\[
\int_{0.05}^{0.15} 150x \, dx = 75x^2 \bigg|_{0.05}^{0.15} = 1.5 \text{ J}.
\]

6. Compressing the spring 4 more cm when it is already compressed 5 cm

SOLUTION The work required to compress the spring from 5 cm to 9 cm past equilibrium is

\[
\int_{-0.05}^{-0.09} 150x \, dx = 75x^2 \bigg|_{-0.05}^{-0.09} = 0.42 \text{ J}.
\]

7. If 5 J of work are needed to stretch a spring 10 cm beyond equilibrium, how much work is required to stretch it 15 cm beyond equilibrium?

SOLUTION First, we determine the value of the spring constant as follows:

\[
\int_{0}^{0.1} kx \, dx = \frac{1}{2} kx^2 \bigg|_{0}^{0.1} = 0.005k = 5 \text{ J}.
\]

Thus, \( k = 1000 \text{ kg/s}^2 \). Next, we calculate the work required to stretch the spring 15 cm beyond equilibrium:

\[
\int_{0}^{0.15} 1000x \, dx = 500x^2 \bigg|_{0}^{0.15} = 11.25 \text{ J}.
\]

8. If 5 J of work are needed to stretch a spring 10 cm beyond equilibrium, how much work is required to compress it 5 cm beyond equilibrium?

SOLUTION First, we determine the value of the spring constant as follows:

\[
\int_{0}^{0.1} kx \, dx = \frac{1}{2} kx^2 \bigg|_{0}^{0.1} = 0.005k = 5 \text{ J}.
\]

Thus, \( k = 1000 \text{ kg/s}^2 \). Next, we calculate the work required to compress the spring 5 cm beyond equilibrium:

\[
\int_{0}^{-0.05} 1000x \, dx = 500x^2 \bigg|_{0}^{-0.05} = 1.25 \text{ J}.
\]

9. If 10 ft-lb of work are needed to stretch a spring 1 ft beyond equilibrium, how far will the spring stretch if a 10-lb weight is attached to its end?

SOLUTION First, we determine the value of the spring constant as follows:

\[
\int_{0}^{1} kx \, dx = \frac{1}{2} kx^2 \bigg|_{0}^{1} = \frac{1}{2} k = 10 \text{ ft-lb}.
\]

Thus \( k = 20 \text{ lb/ft} \). Balancing the forces acting on the weight, we have 10 lb = \( kd = 20d \), which implies \( d = 0.5 \text{ ft} \). A 10-lb weight will therefore stretch the spring 6 inches.

10. Show that the work required to stretch a spring from position \( a \) to position \( b \) is \( \frac{1}{2} k(b^2 - a^2) \), where \( k \) is the spring constant. How do you interpret the negative work obtained when \( |b| < |a| \)?

SOLUTION The work required to stretch a spring from position \( a \) to position \( b \) is

\[
\int_{a}^{b} kx \, dx = \frac{1}{2} kx^2 \bigg|_{a}^{b} = \frac{1}{2} k(b^2 - a^2).
\]

When \( |b| < |a| \), the “negative work” is the work done by the spring to return to its equilibrium position.

In Exercises 11–14, calculate the work against gravity required to build the structure out of brick using the method of Examples 2 and 3. Assume that brick has density 80 lb/ft³.
11. A tower of height 20 ft and square base of side 10 ft

**SOLUTION** The volume of one layer is \(100 \Delta y \text{ ft}^3\) and so the weight of one layer is \(8000 \Delta y \text{ lb}\). Thus, the work done against gravity to build the tower is

\[
W = \int_0^{20} 8000y \, dy = 4000y^2 \bigg|_0^{20} = 1.6 \times 10^6 \text{ ft-lb}.
\]

12. A cylindrical tower of height 20 ft and radius 10 ft

**SOLUTION** The area of the base is \(100\pi \text{ ft}^2\), so the volume of each small layer is \(100\pi \Delta y \text{ ft}^3\). The weight of one layer is then \(8000\pi \Delta y \text{ lb}\). Finally, the total work done against gravity to build the tower is

\[
\int_0^{20} 8000\pi y \, dy = 1.6 \times 10^6 \pi \text{ ft-lb}.
\]

13. A 20-ft-high tower in the shape of a right circular cone with base of radius 4 ft

**SOLUTION** From similar triangles, the area of one layer is \(\pi \left(4 - \frac{y}{5}\right)^2 \text{ ft}^2\), so the volume of each small layer is \(\pi \left(4 - \frac{y}{5}\right)^2 \Delta y \text{ ft}^3\). The weight of one layer is then \(80\pi \left(4 - \frac{y}{5}\right)^2 \Delta y \text{ lb}\). Finally, the total work done against gravity to build the tower is

\[
\int_0^{20} 80\pi \left(4 - \frac{y}{5}\right)^2 \, dy = \frac{128,000\pi}{3} \text{ ft-lb}.
\]

14. A structure in the shape of a hemisphere of radius 4 ft

**SOLUTION** The area of one layer is \(\pi \left(16 - y^2\right) \text{ ft}^2\), so the volume of each small layer is \(\pi \left(16 - y^2\right) \Delta y \text{ ft}^3\). The weight of one layer is then \(80\pi \left(16 - y^2\right) \Delta y \text{ lb}\). Finally, the total work done against gravity to build the tower is

\[
\int_0^4 80\pi \left(16 - y^2\right) y \, dy = 5120\pi \text{ ft-lb}.
\]

15. Built around 2600 BCE, the Great Pyramid of Giza in Egypt is 485 ft high (due to erosion, its current height is slightly less) and has a square base of side 755.5 ft (Figure 6). Find the work needed to build the pyramid if the density of the stone is estimated at 125 lb/ft$^3$.

**FIGURE 6** The Great Pyramid in Giza, Egypt.

**SOLUTION** From similar triangles, the area of one layer is

\[
\left(755.5 - \frac{755.5}{485} y\right)^2 \text{ ft}^2,
\]

so the volume of each small layer is

\[
\left(755.5 - \frac{755.5}{485} y\right)^2 \Delta y \text{ ft}^3.
\]

The weight of one layer is then

\[
125 \left(755.5 - \frac{755.5}{485} y\right)^2 \Delta y \text{ lb}.
\]

Finally, the total work needed to build the pyramid was

\[
\int_0^{485} 125 \left(755.5 - \frac{755.5}{485} y\right)^2 y \, dy = 1.399 \times 10^{12} \text{ ft-lb}.
\]
In Exercises 16–20, calculate the work (in joules) required to pump all of the water out of the tank. Assume that the tank is full, distances are measured in meters, and the density of water is 1,000 kg/m³.

16. The box in Figure 7; water exits from a small hole at the top.

![Figure 7](image)

**SOLUTION** Place the origin on the top of the box, and let the positive y-axis point downward. The volume of one layer of water is $32\Delta y$ m³, so the force needed to lift each layer is

$$(9.8)(1000)32\Delta y = 313600\Delta y \text{ N}.$$  

Each layer must be lifted $y$ meters, so the total work needed to empty the tank is

$$
\int_0^5 313600y \, dy = 156800y^2\bigg|_0^5 = 3.92 \times 10^6 \text{ J}.
$$

17. The hemisphere in Figure 8; water exits from the spout as shown.

![Figure 8](image)

**SOLUTION** Place the origin at the center of the hemisphere, and let the positive y-axis point downward. The radius of a layer of water at depth $y$ is $\sqrt{100 - y^2}$ m, so the volume of the layer is $\pi(100 - y^2)\Delta y$ m³, and the force needed to lift the layer is $9800\pi(100 - y^2)\Delta y$ N. The layer must be lifted $y + 2$ meters, so the total work needed to empty the tank is

$$
\int_0^{10} 9800\pi(100 - y^2)(y + 2) \, dy = \frac{112700000\pi}{3} \approx 1.18 \times 10^8 \text{ J}.
$$

18. The conical tank in Figure 9; water exits through the spout as shown.

![Figure 9](image)

**SOLUTION** Place the origin at the vertex of the inverted cone, and let the positive y-axis point upward. Consider a layer of water at a height of $y$ meters. From similar triangles, the area of the layer is

$$
\pi \left( \frac{y}{2} \right)^2 \text{ m}^2,
$$

so the volume is

$$
\pi \left( \frac{y}{2} \right)^2 \Delta y \text{ m}^3.
$$

Thus the weight of one layer is

$$
9800\pi \left( \frac{y}{2} \right)^2 \Delta y \text{ N}.
$$

The layer must be lifted $12 - y$ meters, so the total work needed to empty the tank is

$$
\int_0^{10} 9800\pi \left( \frac{y}{2} \right)^2 (12 - y) \, dy = \pi(3.675 \times 10^6) \approx 1.155 \times 10^7 \text{ J}.
$$
19. The horizontal cylinder in Figure 10; water exits from a small hole at the top. *Hint:* Evaluate the integral by interpreting part of it as the area of a circle.

![Figure 10](image)

**SOLUTION** Place the origin along the axis of the cylinder. At location \(y\), the layer of water is a rectangular slab of length \(\ell\), width \(2\sqrt{r^2 - y^2}\) and thickness \(\Delta y\). Thus, the volume of the layer is \(2\ell \sqrt{r^2 - y^2} \Delta y\), and the force needed to lift the layer is \(19600\ell \sqrt{r^2 - y^2} \Delta y\). The layer must be lifted a distance \(r - y\), so the total work needed to empty the tank is given by

\[
\int_{-r}^{r} 19600\ell \sqrt{r^2 - y^2} (r - y) \, dy = 19600\ell r \int_{-r}^{r} \sqrt{r^2 - y^2} \, dy - 19600\ell \int_{-r}^{r} y \sqrt{r^2 - y^2} \, dy.
\]

Now,

\[
\int_{-r}^{r} y \sqrt{r^2 - y^2} \, du = 0
\]

because the integrand is an odd function and the integration interval is symmetric with respect to zero. Moreover, the other integral is one-half the area of a circle of radius \(r\); thus,

\[
\int_{-r}^{r} \sqrt{r^2 - y^2} \, dy = \frac{1}{2} \pi r^2.
\]

Finally, the total work needed to empty the tank is

\[
19600\ell r \left( \frac{1}{2} \pi r^2 \right) - 19600\ell (0) = 9800\ell \pi r^3 \text{ J}.
\]

20. The trough in Figure 11; water exits by pouring over the sides.

![Figure 11](image)

**SOLUTION** Place the origin along the bottom edge of the trough, and let the positive \(y\)-axis point upward. From similar triangles, the width of a layer of water at a height of \(y\) meters is

\[
w = a + \frac{y (b - a)}{h} \text{ m}^2,
\]

so the volume of each layer is

\[
c \left( a + \frac{y (b - a)}{h} \right) \Delta y \text{ m}^3.
\]

Thus, the force needed to lift the layer is

\[
9800c \left( a + \frac{y (b - a)}{h} \right) \Delta y \text{ N}.
\]

Each layer must be lifted \(h - y\) meters, so the total work needed to empty the tank is

\[
\int_{0}^{h} 9800(h - y)c \left( a + \frac{y (b - a)}{h} \right) \, dy = 9800c \left( \frac{ah^2}{3} + \frac{bh^2}{6} \right) \text{ J}.
\]

21. Find the work \(W\) required to empty the tank in Figure 7 if it is half full of water.
SOLUTION Place the origin on the top of the box, and let the positive $y$-axis point downward. Note that with this coordinate system, the bottom half of the box corresponds to $y$ values from 2.5 to 5. The volume of one layer of water is $32\Delta y \text{ m}^3$, so the force needed to lift each layer is

$$ (9.8)(1000)32\Delta y = 313600\Delta y \text{ N}. $$

Each layer must be lifted $y$ meters, so the total work needed to empty the tank is

$$ \int_{2.5}^{5} 313600y \, dy = 156800y^2 \bigg|_{2.5}^{5} = 2.94 \times 10^6 \text{ J}. $$

22. Assume the tank in Figure 7 is full of water and let $W$ be the work required to pump out half of the water. Do you expect $W$ to equal the work computed in Exercise 21? Explain and then compute $W$.

SOLUTION Recall that the origin was placed at the top of the box with the positive $y$-axis pointing downward. Pumping out half the water from a full tank would involve $y$ values ranging from $y = 0$ to $y = 2.5$, whereas pumping out a half-full tank would involve $y$ values ranging from $y = 2.5$ to $y = 5$. Because pumping out half the water from a full tank requires moving the layers of water a shorter distance than pumping out a half-full tank, we do not expect that $W$ would be equal to the work computed in Exercise 21.

To compute $W$, we proceed as in Exercise 16 and Exercise 21, to find

$$ W = \int_{0}^{2.5} 313600y \, dy = 980,000 \text{ J}. $$

It is reassuring to note that

$$ \text{Work(Exercise 21)} + \text{Work(Exercise 22)} = \text{Work(Exercise 16)}. $$

23. Find the work required to empty the tank in Figure 9 if it is half full of water.

SOLUTION Place the origin at the vertex of the inverted cone, and let the positive $y$-axis point upward. Consider a layer of water at a height of $y$ meters. From similar triangles, the area of the layer is

$$ \pi \left( \frac{y}{2} \right)^2 \text{ m}^2, $$

so the volume is

$$ \pi \left( \frac{y}{2} \right)^2 \Delta y \text{ m}^3. $$

Thus the weight of one layer is

$$ 9800\pi \left( \frac{y}{2} \right)^2 \Delta y \text{ N}. $$

The layer must be lifted $12 - y$ meters, so the total work needed to empty the half-full tank is

$$ \int_{0}^{5} 9800\pi \left( \frac{y}{2} \right)^2 (12 - y) \, dy = \frac{1684375\pi}{2} \approx 2.65 \times 10^6 \text{ J}. $$

24. Assume the tank in Figure 9 is full of water and find the work required to pump out half of the water.

SOLUTION Place the origin at the vertex of the inverted cone, and let the positive $y$-axis point upward. Note that with this coordinate system, the water in the top half of the cone corresponds to $y$ values ranging from $y = 5$ to $y = 10$. Now, consider a layer of water at a height of $y$ meters. From similar triangles, the area of the layer is

$$ \pi \left( \frac{y}{2} \right)^2 \text{ m}^2, $$

so the volume is

$$ \pi \left( \frac{y}{2} \right)^2 \Delta y \text{ m}^3. $$

Thus the weight of one layer is

$$ 9800\pi \left( \frac{y}{2} \right)^2 \Delta y \text{ N}. $$

The layer must be lifted $12 - y$ meters, so the total work needed to empty the half-full tank is

$$ \int_{5}^{10} 9800\pi \left( \frac{y}{2} \right)^2 (12 - y) \, dy = \frac{5665625\pi}{2} \approx 8.90 \times 10^6 \text{ J}. $$
25. Assume that the tank in Figure 9 is full.
(a) Calculate the work $F(y)$ required to pump out water until the water level has reached level $y$.
(b) CAS: Plot $F(y)$.
(c) CAS: What is the significance of $F'(y)$ as a rate of change?
(d) CAS: If your goal is to pump out all of the water, at which water level $y_0$ will half of the work be done?

**SOLUTION**
(a) Place the origin at the vertex of the inverted cone, and let the positive $y$-axis point upward. Consider a layer of water at a height of $y$ meters. From similar triangles, the area of the layer is
$$
\pi \left( \frac{y}{2} \right)^2 \text{ m}^2,
$$
so the volume is
$$
\pi \left( \frac{y}{2} \right)^2 \Delta y \text{ m}^3.
$$
Thus the weight of one layer is
$$
9800\pi \left( \frac{y}{2} \right)^2 \Delta y \text{ N}.
$$
The layer must be lifted $12 - y$ meters, so the total work needed to pump out water until the water level has reached level $y$ is
$$
\int_y^{10} 9800\pi \left( \frac{y}{2} \right)^2 (12 - y) \, dy = 3675000\pi - 9800\pi y^3 + \frac{1225\pi}{2} y^4 \text{ J}.
$$
(b) A plot of $F(y)$ is shown below.

(c) First, note that $F'(y) < 0$; as $y$ increases, less water is being pumped from the tank, so $F(y)$ decreases. Therefore, when the water level in the tank has reached level $y$, we can interpret $-F'(y)$ as the amount of work per meter needed to remove the next layer of water from the tank. In other words, $-F'(y)$ is a “marginal work” function.
(d) The amount of work needed to empty the tank is $3675000\pi$ J. Half of this work will be done when the water level reaches height $y_0$ satisfying
$$
3675000\pi - 9800\pi y_0^3 + \frac{1225\pi}{2} y_0^4 = 1837500\pi.
$$
Using a computer algebra system, we find $y_0 = 6.91$ m.

26. How much work is done lifting a 25-ft chain over the side of a building (Figure 12)? Assume that the chain has a density of 4 lb/ft. *Hint:* Break up the chain into $N$ segments, estimate the work performed on a segment, and compute the limit as $N \to \infty$ as an integral.

**FIGURE 12** The small segment of the chain of length $\Delta y$ located $y$ feet from the top is lifted through a vertical distance $y$. 
In this example, each part of the chain is moved a different distance. Therefore, we divide the chain into \( N \) small segments of length \( \Delta y = 25/N \). Suppose that the \( i \)th segment is located a distance \( y_i \) from the top of the building. This segment weighs \( 4\Delta y \) pounds and it must be moved approximately \( y_i \) feet (not exactly \( y_i \) feet, because each point along the segment is a slightly different distance from the top). The work \( W_i \) done on this segment is approximately \( W_i \approx 4y_i \Delta y \) ft-lb. The total work \( W \) is the sum of the \( W_i \) and we have

\[
W = \sum_{j=1}^{N} W_j \approx \sum_{j=1}^{N} 4y_j \Delta y.
\]

Passing to the limit as \( N \to \infty \), we obtain

\[
W = \int_{0}^{25} 4 \, dy = 2y^2 \bigg|_{0}^{25} = 1250 \text{ ft-lb}.
\]

27. How much work is done lifting a 3-m chain over the side of a building if the chain has mass density 4 kg/m?

**SOLUTION** Consider a segment of the chain of length \( \Delta y \) located a distance \( y_j \) meters from the top of the building. The work needed to lift this segment of the chain to the top of the building is approximately

\[
W_j \approx (4\Delta y)(9.8)y_j \text{ J}.
\]

Summing over all segments of the chain and passing to the limit as \( \Delta y \to 0 \), it follows that the total work is

\[
\int_{0}^{3} 4 \cdot 9.8y \, dy = 19.6y^2 \bigg|_{0}^{3} = 176.4 \text{ J}.
\]

28. An 8-ft chain weighs 16 lb. Find the work required to lift the chain over the side of a building.

**SOLUTION** First, note that the chain has a mass density of 2 lb/ft. Now, consider a segment of the chain of length \( \Delta y \) located a distance \( y_j \) feet from the top of the building. The work needed to lift this segment of the chain to the top of the building is approximately

\[
W_j \approx (2\Delta y)y_j \text{ ft-lb}.
\]

Summing over all segments of the chain and passing to the limit as \( \Delta y \to 0 \), it follows that the total work is

\[
\int_{0}^{8} 2y \, dy = y^2 \bigg|_{0}^{8} = 64 \text{ ft-lb}.
\]

29. A 20-ft chain with mass density 3 lb/ft is initially coiled on the ground. How much work is performed in lifting the chain so that it is fully extended (and one end touches the ground)?

**SOLUTION** Consider a segment of the chain of length \( \Delta y \) that must be lifted \( y_j \) feet off the ground. The work needed to lift this segment of the chain is approximately

\[
W_j \approx (3\Delta y)y_j \text{ ft-lb}.
\]

Summing over all segments of the chain and passing to the limit as \( \Delta y \to 0 \), it follows that the total work is

\[
\int_{0}^{20} 3y \, dy = \frac{3}{2}y^2 \bigg|_{0}^{20} = 600 \text{ ft-lb}.
\]

30. How much work is done lifting a 20-ft chain with mass density 3 lb/ft (initially coiled on the ground) so that its top end is 30 ft above the ground?

**SOLUTION** After the chain has been lifted 20 ft, the problem becomes one of simply lifting a weight 10 ft above the ground. The weight of the chain is 60 lb. Thus, the total work is

\[
\int_{0}^{20} 3y \, dy + \int_{0}^{10} 60 \, dy = \frac{3}{2}y^2 \bigg|_{0}^{20} + 60y \bigg|_{0}^{10} = 1200 \text{ ft-lb}.
\]

31. A 1,000-lb wrecking ball hangs from a 30-ft cable of density 10 lb/ft attached to a crane. Calculate the work done if the crane lifts the ball from ground level to 30 ft in the air by drawing in the cable.
**APPLICATIONS OF THE INTEGRAL**

SOLUTION We will treat the cable and the wrecking ball separately. Consider a segment of the cable of length $\Delta y$ that must be lifted $y_j$ feet. The work needed to lift the cable segment is approximately

$$W_j \approx (10\Delta y)y_j \text{ ft-lb}.$$  

Summing over all of the segments of the cable and passing to the limit as $\Delta y \to 0$, it follows that lifting the cable requires

$$\int_0^{30} 10y \, dy = 5y^2 \bigg|_0^{30} = 4500 \text{ ft-lb}.$$  

Lifting the 1000 lb wrecking ball 30 feet requires an additional 30,000 ft-lb. Thus, the total work is 34,500 ft-lb.

In Exercises 32–34, use Newton’s Universal Law of Gravity, according to which the gravitational force between two objects of mass $m$ and $M$ separated by a distance $r$ has magnitude $GMm/r^2$, where $G = 6.67 \times 10^{-11} \text{ m}^3 \text{kg}^{-1} \text{s}^{-1}$. Although the Universal Law refers to point masses, Newton proved that it also holds for uniform spherical objects, where $r$ is the distance between their centers.

32. Two spheres of mass $M$ and $m$ are separated by a distance $r_1$. Show that the work required to increase the separation to a distance $r_2$ is equal to $W = GMm(r_1^{-1} - r_2^{-1})$.

**SOLUTION** The work required to increase the separation from a distance $r_1$ to a distance $r_2$ is

$$\int_{r_1}^{r_2} \frac{GMm}{r^2} \, dr = -\frac{GMm}{r} \bigg|_{r_1}^{r_2} = GMm(r_1^{-1} - r_2^{-1}).$$

33. Use the result of Exercise 32 to calculate the work required to place a 2,000-kg satellite in an orbit 1,200 km above the surface of the earth. Assume that the earth is a sphere of mass $M_e = 5.98 \times 10^{24} \text{ kg}$ and radius $r_e = 6.37 \times 10^6 \text{ m}$. Treat the satellite as a point mass.

**SOLUTION** The satellite will move from a distance $r_1 = r_e$ to a distance $r_2 = r_e + 1200000$. Thus, from Exercise 32,

$$W = (6.67 \times 10^{-11})(5.98 \times 10^{24})(2000) \left( \frac{1}{6.37 \times 10^6} - \frac{1}{6.37 \times 10^6 + 1200000} \right) \approx 1.99 \times 10^{10} \text{ J}.$$  

34. Use the result of Exercise 32 to compute the work required to move a 1,500-kg satellite from an orbit 1,000 to 1,500 km above the surface of the earth.

**SOLUTION** The satellite will move from a distance $r_1 = r_e + 1000000$ to a distance $r_2 = r_e + 1500000$. Thus, from Exercise 32,

$$W = (6.67 \times 10^{-11})(5.98 \times 10^{24})(1500) \times \left( \frac{1}{6.37 \times 10^6 + 1000000} - \frac{1}{6.37 \times 10^6 + 1500000} \right) \approx 5.16 \times 10^9 \text{ J}.$$  

35. Assume that the pressure $P$ and volume $V$ of the gas in a 30-in. cylinder of radius 3 in. with a movable piston are related by $PV^{1.4} = k$, where $k$ is a constant (Figure 13). When the cylinder is full, the gas pressure is 200 lb/in.$^2$.

(a) Calculate $k$.

(b) Calculate the force on the piston as a function of the length $x$ of the column of gas (the force is $PA$, where $A$ is the piston’s area).

(c) Calculate the work required to compress the gas column from 30 to 20 in.

![Gas in a cylinder with a piston](image)

**FIGURE 13** Gas in a cylinder with a piston.

**(a) We have** $P = 200$ and $V = 270\pi$. Thus

$$k = 200(270\pi)^{1.4} = 2.517382 \times 10^6 \text{ lb-in}^{2.2}.$$  

**(b) The area of the piston is** $A = 9\pi$ and the volume of the cylinder as a function of $x$ is $V = 9\pi x$, which gives $P = k/V^{1.4} = k/(9\pi x)^{1.4}$. Thus

$$F = PA = \frac{k}{(9\pi x)^{1.4}}9\pi = k(9\pi)^{-0.4}x^{-1.4}.$$
(c) Since the force is pushing against the piston, in order to calculate work, we must calculate the integral of the opposite force, i.e., we have
\[
W = -k(9\pi)^{-0.4} \int_{30}^{20} x^{-1.4} \, dx = -k(9\pi)^{-0.4} \left. \frac{1}{-0.4} x^{-0.4} \right|_{30}^{20} = 74777.8 \text{ in-lb}.
\]

**Further Insights and Challenges**

36. A 20-ft chain with linear mass density
\[
\rho(x) = 0.02x(20 - x) \text{ lb/ft}
\]
lies on the ground.

(a) How much work is done lifting the chain so that it is fully extended (and one end touches the ground)?

(b) How much work is done lifting the chain so that its top end has a height of 30 ft?

**SOLUTION**

(a) Consider a segment of the chain of length $\Delta x$ that must be lifted $x_j$ feet. The work needed to lift this segment is approximately
\[
W_j \approx (\rho(x_j) \Delta x) x_j \text{ ft-lb}.
\]

Summing over all segments of the chain and passing to the limit as $\Delta x \to 0$, it follows that the total work is
\[
\int_{0}^{20} \rho(x) x \, dx = \int_{0}^{20} \left(0.4x^2 - 0.02x^3\right) \, dx = \left(\frac{4}{3} x^3 - \frac{0.02}{4} x^4\right) \bigg|_{0}^{20} = \frac{800}{3} \text{ ft-lb}.
\]

(b) From part (a), lifting the chain so that it is fully extended requires $\frac{800}{3}$ ft-lb of work. Lifting the entire chain, which weighs another ten feet requires an additional $\frac{800}{3}$ ft-lb of work. The total work is therefore $\frac{1600}{3}$ ft-lb.

37. **Work-Kinetic Energy Theorem**

The kinetic energy of an object of mass $m$ moving with velocity $v$ is $KE = \frac{1}{2}mv^2$.

(a) Suppose that the object moves from $x_1$ to $x_2$ during the time interval $[t_1, t_2]$ due to a net force $F(x)$ acting along the interval $[x_1, x_2]$. Let $x(t)$ be the position of the object at time $t$. Use the Change of Variables formula to show that the work performed is equal to
\[
W = \int_{x_1}^{x_2} F(x) \, dx = \int_{t_1}^{t_2} F(x(t))v(t) \, dt
\]

(b) By Newton’s Second Law, $F(x(t)) = ma(t)$, where $a(t)$ is the acceleration at time $t$. Show that
\[
\frac{d}{dt} \left(\frac{1}{2}mv(t)^2\right) = F(x(t))v(t)
\]

(c) Use the FTC to show that the change in kinetic energy during the time interval $[t_1, t_2]$ is equal to
\[
\int_{t_1}^{t_2} F(x(t))v(t) \, dt.
\]

(d) Prove the Work-Kinetic Energy Theorem: The change in KE is equal to the work $W$ performed.

**SOLUTION**

(a) Let $x_1 = x(t_1)$ and $x_2 = x(t_2)$, then $x = x(t)$ gives $dx = v(t) \, dt$. By substitution we have
\[
W = \int_{x_1}^{x_2} F(x) \, dx = \int_{t_1}^{t_2} F(x(t))v(t) \, dt.
\]

(b) Knowing $F(x(t)) = m \cdot a(t)$, we have
\[
\frac{d}{dt} \left(\frac{1}{2}mv(t)^2\right) = m \cdot v(t) v'(t) \quad \text{(Chain Rule)}
\]
\[
= m \cdot v(t) a(t)
\]
\[
= v(t) \cdot F(x(t)) \quad \text{(Newton’s 2nd law)}
\]
(c) From the FTC, 
\[ \frac{1}{2} m \cdot v(t)^2 = \int F(x(t)) \ v(t) \ dt. \]

Since \( KE = \frac{1}{2} m v^2 \), 
\[ \Delta KE = KE(t_2) - KE(t_1) = \frac{1}{2} m v(t_2)^2 - \frac{1}{2} m v(t_1)^2 = \int_{t_1}^{t_2} F(x(t)) \ v(t) \ dt. \]

(d) 
\[ W = \int_{x_1}^{x_2} F(x) \ dx = \int_{t_1}^{t_2} F(x(t)) \ v(t) \ dt \quad \text{(Part (a))} \]
\[ = KE(t_2) - KE(t_1) \quad \text{(Part (c))} \]
\[ = \Delta KE \quad \text{(as required)} \]

38. A model train of mass 0.5 kg is placed at one end of a straight 3-m electric track. Assume that a force \( F(x) = 3x - x^2 \) N acts on the train at distance \( x \) along the track. Use the Work-Kinetic Energy Theorem (Exercise 37) to determine the velocity of the train when it reaches the end of the track.

**SOLUTION** We have 
\[ W = \int_0^3 (3x - x^2) \ dx = \left[ \frac{3}{2} x^2 - \frac{1}{3} x^3 \right]_0^3 = 4.5 \text{ J}. \]

Then the change in KE must be equal to \( W \), which gives 
\[ 4.5 = \frac{1}{2} m (v(t_2)^2 - v(t_1)^2). \]

Note that \( v(t_1) = 0 \) as the train was placed on the track with no initial velocity and \( m = .5 \). Thus 
\[ v(t_2) = \sqrt{18} = 4.242641 \text{ m/sec}. \]

39. With what initial velocity \( v_0 \) must we fire a rocket so it attains a maximum height \( r \) above the earth? *Hint:* Use the results of Exercises 32 and 37. As the rocket reaches its maximum height, its KE decreases from \( \frac{1}{2} m v_0^2 \) to zero.

**SOLUTION** The work required to move the rocket a distance \( r \) from the surface of the earth is 
\[ W(r) = GM_e m \left( \frac{1}{r_e} - \frac{1}{r + r_e} \right). \]

As the rocket climbs to a height \( r \), its kinetic energy is reduced by the amount \( W(r) \). The rocket reaches its maximum height when its kinetic energy is reduced to zero, that is, when 
\[ \frac{1}{2} m v_0^2 = GM_e m \left( \frac{1}{r_e} - \frac{1}{r + r_e} \right). \]

Therefore, its initial velocity must be 
\[ v_0 = \sqrt{2 G M_e \left( \frac{1}{r_e} - \frac{1}{r + r_e} \right)}. \]

40. With what initial velocity must we fire a rocket so it attains a maximum height of \( r = 20 \) km above the surface of the earth?

**SOLUTION** Using the result of the previous exercise with \( G = 6.67 \times 10^{-11} \text{ m}^3 \text{kg}^{-1} \text{s}^{-2}, M_e = 5.98 \times 10^{24} \text{ kg}, r_e = 6.37 \times 10^6 \text{ m} \) and \( r = 20000 \text{ m} \), 
\[ v_0 = \sqrt{2 G M_e \left( \frac{1}{r_e} - \frac{1}{r + r_e} \right)} = 626 \text{ m/sec}. \]

41. Calculate escape velocity, the minimum initial velocity of an object to ensure that it will continue traveling into space and never fall back to earth (assuming that no force is applied after takeoff). *Hint:* Take the limit as \( r \to \infty \) in Exercise 39.
SOLUTION The result of the previous exercise leads to an interesting conclusion. The initial velocity $v_0$ required to reach a height $r$ does not increase beyond all bounds as $r$ tends to infinity; rather, it approaches a finite limit, called the escape velocity:

$$v_{\text{esc}} = \lim_{r \to \infty} \sqrt{\frac{2GM_e}{r_e} \left( \frac{1}{r} - \frac{1}{r+e} \right)} = \sqrt{\frac{2GM_e}{r_e}}$$

In other words, $v_{\text{esc}}$ is large enough to insure that the rocket reaches a height $r$ for every value of $r$! Therefore, a rocket fired with initial velocity $v_{\text{esc}}$ never returns to earth. It continues traveling indefinitely into outer space.

Now, let’s see how large escape velocity actually is:

$$v_{\text{esc}} = \left( \frac{2 \cdot 6.67 \times 10^{-11} \cdot 5.989 \times 10^{24}}{6.37 \times 10^6} \right)^{1/2} \approx 11,190 \text{ m/sec.}$$

Since one meter per second is equal to 2.236 miles per hour, escape velocity is approximately $11,190(2.236) = 25,020$ miles per hour.

### CHAPTER REVIEW EXERCISES

In Exercises 1–6, find the area of the region bounded by the graphs of the functions.

1. $y = \sin x$, $y = \cos x$, $0 \leq x \leq \frac{5\pi}{4}$

SOLUTION The region bounded by the graphs of $y = \sin x$ and $y = \cos x$ over the interval $[0, \frac{5\pi}{4}]$ is shown below. For $x \in [0, \frac{\pi}{4}]$, the graph of $y = \cos x$ lies above the graph of $y = \sin x$, whereas, for $x \in [\frac{\pi}{4}, \frac{5\pi}{4}]$, the graph of $y = \sin x$ lies above the graph of $y = \cos x$. The area of the region is therefore given by

$$\int_0^{\pi/4} (\cos x - \sin x) \, dx + \int_{\pi/4}^{5\pi/4} (\sin x - \cos x) \, dx$$

$$= (\sin x + \cos x) \bigg|_0^{\pi/4} + (-\cos x - \sin x) \bigg|_{\pi/4}^{5\pi/4}$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - (0 + 1) + \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) = 3\sqrt{2} - 1.$$  

![Graph of y = sin x and y = cos x](image-url)

2. $f(x) = x^3 - 2x^2 + x$, $g(x) = x^2 - x$

SOLUTION The region bounded by the graphs of $y = x^3 - 2x^2 + x$ and $y = x^2 - x$ over the interval $[0, 2]$ is shown below. For $x \in [0, 1]$, the graph of $y = x^3 - 2x^2 + x$ lies above the graph of $y = x^2 - x$, whereas, for $x \in [1, 2]$, the graph of $y = x^2 - x$ lies above the graph of $y = x^3 - 2x^2 + x$. The area of the region is therefore given by

$$\int_0^1 (x^3 - 2x^2 + x) - (x^2 - x) \, dx + \int_1^2 (x^2 - x) - (x^3 - 2x^2 + x) \, dx$$

$$= \left(\frac{1}{4}x^4 - x^3 + x^2\right) \bigg|_0^1 + \left(x^3 - x^2 - \frac{1}{4}x^4\right) \bigg|_1^2$$

$$= \frac{1}{4} - 1 + 1 + (8 - 4 - 4) - \left(1 - 1 - \frac{1}{4}\right) = \frac{1}{2}.$$  

![Graph of y = sin x and y = cos x](image-url)
3. $f(x) = x^2 + 2x, \quad g(x) = x^2 - 1, \quad h(x) = x^2 + x - 2$

**SOLUTION** The region bounded by the graphs of $y = x^2 + 2x$, $y = x^2 - 1$ and $y = x^2 + x - 2$ is shown below. For each $x \in [-2, -\frac{1}{2}]$, the graph of $y = x^2 + 2x$ lies above the graph of $y = x^2 + x - 2$, whereas, for each $x \in [-\frac{1}{2}, 1]$, the graph of $y = x^2 - 1$ lies above the graph of $y = x^2 + x - 2$. The area of the region is therefore given by

$$
\int_{-2}^{-1/2} \left( (x^2 + 2x) - (x^2 + x - 2) \right) \, dx + \int_{-1/2}^{1} \left( (x^2 - 1) - (x^2 + x - 2) \right) \, dx \\
= \left[ \frac{1}{2} x^2 + 2x \right]_{-2}^{-1/2} + \left[ -\frac{1}{2} x^2 + x \right]_{-1/2}^{1} \\
= \left( \frac{1}{2} - 1 \right) - (2 - 4) + \left( -\frac{1}{2} + 1 \right) - \left( -\frac{1}{8} - \frac{1}{2} \right) = \frac{9}{4}.
$$

4. $f(x) = \sin x, \quad g(x) = \sin 2x, \quad \frac{\pi}{3} \leq x \leq \pi$

**SOLUTION** The region bounded by the graphs of $y = \sin x$ and $y = \sin 2x$ over the interval $[\frac{\pi}{3}, \pi]$ is shown below. As the graph of $y = \sin x$ lies above the graph of $y = \sin 2x$, the area of the region is given by

$$
\int_{\pi/3}^{\pi} (\sin x - \sin 2x) \, dx = \left[ -\cos x + \frac{1}{2} \cos 2x \right]_{\pi/3}^{\pi} = \left( 1 + \frac{1}{2} \right) - \left( -\frac{1}{2} - \frac{1}{4} \right) = \frac{9}{4}.
$$

5. $y = e^x, \quad y = 1 - x, \quad x = 1$

**SOLUTION** The region bounded by the graphs of $y = e^x$, $y = 1 - x$ and $x = 1$ is shown below. As the graph of $y = e^x$ lies above the graph of $y = 1 - x$, the area of the region is given by

$$
\int_{0}^{1} \left( e^x - (1 - x) \right) \, dx = \left( e^x - x + \frac{1}{2} x^2 \right)_{0}^{1} = \left( e - 1 + \frac{1}{2} \right) - 1 = e - \frac{3}{2}.
$$
6. \( y = \cosh 1 - \cosh x, \quad y = \cosh x - \cosh 1 \)

**SOLUTION** The region bounded by the graphs of \( y = \cosh 1 - \cosh x, \ y = \cosh x - \cosh 1 \) is shown below. As the graph of \( y = \cosh 1 - \cosh x \) lies above the graph of \( y = \cosh x - \cosh 1 \), the area of the region is given by

\[
\int_{-1}^{1} ((\cosh 1 - \cosh x) - (\cosh x - \cosh 1)) \, dx = (2x \cosh 1 - 2 \sinh x) \bigg|_{-1}^{1} = (2 \cosh 1 - 2 \sinh 1) - (-2 \cosh 1 + 2 \sinh 1) = 4 \cosh 1 - 4 \sinh 1 = 4e^{-1}.
\]

In Exercises 7–10, sketch the region bounded by the graphs of the functions and find its area.

7. \( f(x) = x^3 - x^2 - x + 1, \quad g(x) = \sqrt{1 - x^2}, \quad 0 \leq x \leq 1 \)

*Hint:* Use geometry to evaluate the integral.

**SOLUTION** The region bounded by the graphs of \( y = x^3 - x^2 - x + 1 \) and \( y = \sqrt{1 - x^2} \) is shown below. As the graph of \( y = \sqrt{1 - x^2} \) lies above the graph of \( y = x^3 - x^2 - x + 1 \), the area of the region is given by

\[
\int_0^1 (\sqrt{1 - x^2} - (x^3 - x^2 - x + 1)) \, dx.
\]

Now, the region below the graph of \( y = \sqrt{1 - x^2} \) but above the x-axis over the interval \([0, 1]\) is one-quarter of the unit circle; thus,

\[
\int_0^1 \sqrt{1 - x^2} \, dx = \frac{\pi}{4}.
\]

Moreover,

\[
\int_0^1 (x^3 - x^2 - x + 1) \, dx = \left( \frac{1}{4}x^4 - \frac{1}{3}x^3 - \frac{1}{2}x^2 + x \right) \bigg|_{0}^{1} = \frac{1}{4} - \frac{1}{3} - \frac{1}{2} + 1 = \frac{5}{12}.
\]

Finally, the area of the region shown below is

\[
\frac{\pi}{4} - \frac{5}{12}.
\]

8. \( x = \frac{1}{2}y, \quad x = y\sqrt{1 - y^2}, \quad 0 \leq y \leq 1 \)

**SOLUTION** The region bounded by the graphs of \( x = y/2 \) and \( x = y\sqrt{1 - y^2} \) over the interval \([0, 1]\) is shown below. For \( y \in [0, \frac{\sqrt{2}}{2}] \), the graph of \( x = y\sqrt{1 - y^2} \) lies to the right of the graph of \( x = y/2 \), whereas, for \( y \in [\frac{\sqrt{2}}{2}, 1] \), the graph of \( x = y/2 \) lies to the right of the graph of \( x = y\sqrt{1 - y^2} \). The area of the region is therefore given by

\[
\int_0^{\sqrt{3}/2} \left( y\sqrt{1 - y^2} - \frac{y}{2} \right) \, dy + \int_{\sqrt{3}/2}^1 \left( \frac{y}{2} - y\sqrt{1 - y^2} \right) \, dy.
\]
APPLICATIONS OF THE INTEGRAL

1.20.4 0.6 10.2

Use a graphing utility to locate the points of intersection of

\[ y = \sqrt{x}, \quad y = 3x, \quad y = 4 \]

SOLUTION The region bounded by the graphs of \( y = 4 - x^2 \), \( y = 3x \) and \( y = 4 \) is shown below. For \( x \in [0, 1] \), the graph of \( y = 4 \) lies above the graph of \( y = 4 - x^2 \), whereas, for \( x \in [1, \frac{1}{3}] \), the graph of \( y = 4 \) lies above the graph of \( y = 3x \). The area of the region is therefore given by

\[
\int_0^1 (4 - (4 - x^2)) \, dx + \int_{1^{4/3}}^4 (4 - 3x) \, dx = \left. \frac{1}{3} x^3 \right|_0^1 + \left. (4x - \frac{3}{2} x^2) \right|_1^4 = \frac{1}{3} + \left( \frac{16}{3} - \frac{8}{3} \right) - \left( 4 - \frac{3}{2} \right) = \frac{1}{2}.
\]

9. \( y = 4 - x^2, \quad y = 3x, \quad y = 4 \)

10. \( x = y^3 - 2y^2 + y, \quad x = y^2 - y \)

11. [GU] Use a graphing utility to locate the points of intersection of \( y = e^{-x} \) and \( y = 1 - x^2 \) and find the area between the two curves (approximately).

SOLUTION The region bounded by the graphs of \( y = e^{-x} \) and \( y = 1 - x^2 \) is shown below. One point of intersection clearly occurs at \( x = 0 \). Using a computer algebra system, we find that the other point of intersection occurs at \( x = 0.7145563847 \). As the graph of \( y = 1 - x^2 \) lies above the graph of \( y = e^{-x} \), the area of the region is given by

\[
\int_0^{0.7145563847} \left( 1 - x^2 - e^{-x} \right) \, dx = 0.08235024596
\]
12. Figure 1 shows a solid whose horizontal cross section at height \( y \) is a circle of radius \((1 + y)^{-2}\) for \( 0 \leq y \leq H \). Find the volume of the solid.

**SOLUTION** The area of each horizontal cross section is \( A(y) = \pi (1 + y)^{-4} \). Therefore, the volume of the solid is

\[
\int_{0}^{H} \pi (1 + y)^{-4} \, dy = \left[ -\frac{\pi (1 + y)^{-3}}{-3} \right]_{0}^{H} = \pi \left( \frac{(1 + H)^{-3}}{-3} + \frac{1}{3} \right) = \frac{\pi}{3} \left( 1 - \frac{1}{(1 + H)^3} \right).
\]

13. Find the total weight of a 3-ft metal rod of linear density \( \rho(x) = 1 + 2x + \frac{2}{9}x^3 \) lb/ft.

**SOLUTION** The total weight of the rod is

\[
\int_{0}^{3} \rho(x) \, dx = \left[ x + x^2 + \frac{1}{18}x^4 \right]_{0}^{3} = 3 + 9 + \frac{9}{2} = \frac{33}{2} \text{ lb}.
\]

14. Find the flow rate (in the correct units) through a pipe of diameter 6 cm if the velocity of fluid particles at a distance \( r \) from the center of the pipe is \( v(r) = (3 - r) \) cm/s.

**SOLUTION** The flow rate through the pipe is

\[
2\pi \int_{0}^{3} rv(r) \, dr = 2\pi \int_{0}^{3} (3r - r^2) \, dr = 2\pi \left[ \frac{3}{2}r^2 - \frac{1}{3}r^3 \right]_{0}^{3} = 2\pi \left( \frac{27}{2} - 9 \right) = 9\pi \text{ cm}^3/\text{s}.
\]

**In Exercises 15–20, find the average value of the function over the interval.**

15. \( f(x) = x^3 - 2x + 2 \), \([-1, 2]\)

**SOLUTION** The average value is

\[
\frac{1}{2 - (-1)} \int_{-1}^{2} (x^3 - 2x + 2) \, dx = \frac{1}{3} \left[ \frac{1}{4}x^4 - x^2 + 2x \right]_{-1}^{2} = \frac{1}{3} \left[ (4 - 4 + 4) - \left( \frac{1}{4} - 1 - 2 \right) \right] = \frac{9}{4}.
\]

16. \( f(x) = \sqrt{9 - x^2} \), \([0, 3]\) **Hint:** Use geometry to evaluate the integral.

**SOLUTION** The region below the graph of \( y = \sqrt{9 - x^2} \) but above the \( x \)-axis over the interval \([0, 3]\) is one-quarter of a circle of radius 3; consequently,

\[
\int_{0}^{3} \sqrt{9 - x^2} \, dx = \frac{1}{4} \pi (3)^2 = \frac{9\pi}{4}.
\]

The average value is then

\[
\frac{1}{3 - 0} \int_{0}^{3} \sqrt{9 - x^2} \, dx = \frac{1}{3} \left( \frac{9\pi}{4} \right) = \frac{3\pi}{4}.
\]
17. \( f(x) = |x| \), \([-4, 4]\)

**Solution**

The average value is

\[
\frac{1}{4 - (-4)} \int_{-4}^{4} |x| \, dx = \frac{1}{8} \left( \int_{-4}^{0} (-x) \, dx + \int_{0}^{4} x \, dx \right) = \frac{1}{8} \left( -\frac{1}{2}x^2 \bigg|_{-4}^{0} + \frac{1}{2}x^2 \bigg|_{0}^{4} \right) = \frac{1}{8}[(0 + 8) + (8 - 0)] = 2.
\]

18. \( f(x) = x|x| \), \([0, 3]\)

**Solution**

The average value is

\[
\frac{1}{3 - 0} \int_{0}^{3} x|x| \, dx = \frac{1}{3} \left( \int_{0}^{1} x \cdot 0 \, dx + \int_{1}^{2} x \cdot 1 \, dx + \int_{2}^{3} x \cdot 2 \, dx \right) = \frac{1}{3} \left( \frac{1}{2}x^2 \bigg|_{0}^{1} + x^3 \bigg|_{1}^{2} \right) = \frac{1}{3} \left( 2 - \frac{1}{2} + 9 - 4 \right) = \frac{13}{6}.
\]

19. \( f(x) = x \cosh(x^2) \), \([0, 1]\)

**Solution**

The average value is

\[
\frac{1}{1 - 0} \int_{0}^{1} x \cosh(x^2) \, dx.
\]

To evaluate the integral, let \( u = x^2 \). Then \( du = 2x \, dx \) and

\[
\frac{1}{1 - 0} \int_{0}^{1} x \cosh(x^2) \, dx = \frac{1}{2} \int_{0}^{1} \cosh u \, du = \frac{1}{2} \sinh u \bigg|_{0}^{1} = \frac{1}{2} \sinh 1.
\]

20. \( f(x) = \frac{e^x}{1 + e^{2x}} \), \(\left[0, \frac{1}{2}\right]\)

**Solution**

The average value is

\[
\frac{1}{2 - 0} \int_{0}^{1/2} \frac{e^x}{1 + e^{2x}} \, dx.
\]

To evaluate the integral, let \( u = e^x \). Then \( du = e^x \, dx \) and

\[
\frac{1}{2} \int_{0}^{1/2} \frac{e^x}{1 + e^{2x}} \, dx = 2 \int_{1}^{\sqrt{e}} \frac{du}{1 + u^2} = 2 \tan^{-1} u \bigg|_{1}^{\sqrt{e}} = 2 \left( \tan^{-1} \sqrt{e} - \frac{\pi}{4} \right).
\]

21. The average value of \( g(t) \) on \([2, 5]\) is 9. Find \( \int_{2}^{5} g(t) \, dt \).

**Solution**

The average value of the function \( g(t) \) on \([2, 5]\) is given by

\[
\frac{1}{5 - 2} \int_{2}^{5} g(t) \, dt = \frac{1}{3} \int_{2}^{5} g(t) \, dt.
\]

Therefore,

\[
\int_{2}^{5} g(t) \, dt = 3(\text{average value}) = 3(9) = 27.
\]

22. For all \( x \geq 0 \), the average value of \( R(x) \) over \([0, x]\) is equal to \( x \). Find \( R(x) \).

**Solution**

The average value of the function \( R(x) \) over \([0, x]\) is

\[
\frac{1}{x - 0} \int_{0}^{x} R(t) \, dt = \frac{1}{x} \int_{0}^{x} R(t) \, dt.
\]

Given that the average value is equal to \( x \), it follows that

\[
\int_{0}^{x} R(t) \, dt = x^2.
\]

Differentiating both sides of this equation and using the Fundamental Theorem of Calculus on the left-hand side yields

\[
R(x) = 2x.
\]
23. Use the Shell Method to find the volume of the solid obtained by revolving the region between \( y = x^2 \) and \( y = mx \) about the \( x \)-axis (Figure 2).

\[
\int_0^m y \left( \sqrt{y} - \frac{y}{m} \right) \, dy = \left. 2\pi \left( \frac{2}{5} y^{5/2} - \frac{y^3}{3m} \right) \right|_0^m = 2\pi \left( \frac{2m^5}{5} - \frac{m^3}{3} \right) = \frac{\pi}{15} m^5.
\]

24. Use the washer method to find the volume of the solid obtained by revolving the region between \( y = x^2 \) and \( y = mx \) about the \( y \)-axis (Figure 2).

\[
\pi \int_0^m \left( \left( \sqrt{y} \right)^2 - \left( \frac{y}{m} \right)^2 \right) \, dy = \pi \left( \frac{1}{2} y^2 - \frac{y^3}{3m^2} \right) \bigg|_0^m = \pi \left( \frac{m^4}{2} - \frac{m^4}{3} \right) = \pi m^4.
\]

25. Let \( R \) be the intersection of the circles of radius 1 centered at \((1, 0)\) and \((0, 1)\). Express as an integral (but do not evaluate): (a) the area of \( R \) and (b) the volume of revolution of \( R \) about the \( x \)-axis.

\[
\int_0^1 \left( \sqrt{1 - (x - 1)^2} - (1 - \sqrt{1 - x^2}) \right) \, dx.
\]

\[
\pi \int_0^1 \left[ (1 - (x - 1)^2) - (1 - \sqrt{1 - x^2})^2 \right] \, dx.
\]

26. Use the Shell Method to set up an integral (but do not evaluate) expressing the volume of the solid obtained by rotating the region under \( y = \cos x \) over \([0, \pi/2]\) about the line \( x = \pi \).

\[
2\pi \int_0^{\pi/2} (\pi - x) \cos x \, dx.
\]
In Exercises 27–35, find the volume of the solid obtained by rotating the region enclosed by the curves about the given axis.

27. \( y = 2x, \ y = 0, \ x = 8; \ x\)-axis

**SOLUTION** The region bounded by the graphs of \( y = 2x, \ y = 0 \) and \( x = 8 \) is shown below. Let’s choose to slice the region vertically. Because a vertical slice is perpendicular to the axis of rotation, we will use the washer method to calculate the volume of the solid of revolution. For each \( x \in [0, 8] \), the cross section is a circular disk with radius \( R = 2x \). The volume of the solid is therefore given by
\[
\pi \int_0^8 (2x)^2 \, dx = \frac{4\pi}{3} x^3 \bigg|_0^8 = \frac{2048\pi}{3}.
\]

28. \( y = 2x, \ y = 0, \ x = 8; \ \text{axis } x = -3\)

**SOLUTION** Let’s choose to slice the region bounded by the graphs of \( y = 2x, \ y = 0 \) and \( x = 8 \) (see the figure in the previous exercise) vertically. Because a vertical slice is parallel to the axis of rotation, we will use the shell method to calculate the volume of the solid of revolution. For each \( x \in [0, 8] \), the shell has radius \( x - (-3) = x + 3 \) and height \( 2x \). The volume of the solid is therefore given by
\[
2\pi \int_0^8 (x + 3)(2x) \, dx = 4\pi \left(\frac{1}{3} x^3 + \frac{3}{2} x^2\right) \big|_0^8 = 4\pi \left(\frac{512}{3} + 96\right) = \frac{3200\pi}{3}.
\]

29. \( y = x^2 - 1, \ y = 2x - 1, \ \text{axis } x = -2\)

**SOLUTION** The region bounded by the graphs of \( y = x^2 - 1 \) and \( y = 2x - 1 \) is shown below. Let’s choose to slice the region vertically. Because a vertical slice is parallel to the axis of rotation, we will use the shell method to calculate the volume of the solid of revolution. For each \( x \in [0, 2] \), the shell has radius \( x - (-2) = x + 2 \) and height \( (2x - 1) - (x^2 - 1) = 2x - x^2 \). The volume of the solid is therefore given by
\[
2\pi \int_0^2 (x + 2)(2x - x^2) \, dx = 2\pi \left(2x^2 - \frac{1}{4} x^4\right) \big|_0^2 = 2\pi(8 - 4) = 8\pi.
\]

30. \( y = x^2 - 1, \ y = 2x - 1, \ \text{axis } y = 4\)

**SOLUTION** Let’s choose to slice the region bounded by the graphs of \( y = x^2 - 1 \) and \( y = 2x - 1 \) (see the figure in the previous exercise) vertically. Because a vertical slice is perpendicular to the axis of rotation, we will use the washer method to calculate the volume of the solid of revolution. For each \( x \in [0, 2] \), the cross section is a washer with outer radius \( R = 4 - (x^2 - 1) = 5 - x^2 \) and inner radius \( r = 4 - (2x - 1) = 5 - 2x \). The volume of the solid is therefore given by
\[
\pi \int_0^2 \left((5 - x^2)^2 - (5 - 2x)^2\right) \, dx = \pi \left(10x^2 - \frac{14}{3} x^3 + \frac{1}{5} x^5\right) \big|_0^2 = \pi \left(40 - \frac{112}{3} + \frac{32}{5}\right) = \frac{136\pi}{15}.
\]

31. \( y^2 = x^3, \ y = x, \ x = 8; \ \text{axis } x = -1\)
SOLUTION  The region bounded by the graphs of $y^2 = x^3$, $y = x$ and $x = 8$ is composed of two components, shown below. Let's choose to slice the region vertically. Because a vertical slice is parallel to the axis of rotation, we will use the washer method to calculate the volume of the solid of revolution. For each $x \in [0, 1]$, the shell as radius $x - (-1) = x + 1$ and height $x - x^{3/2}$; for each $x \in [1, 8]$, the shell also has radius $x + 1$, but the height is $x^{3/2} - x$. The volume of the solid is therefore given by

$$2\pi \int_0^1 (x + 1)(x - x^{3/2}) \, dx + 2\pi \int_1^8 (x + 1)(x^{3/2} - x) \, dx = \frac{2\pi}{35}(12032\sqrt{2} - 7083).$$

32. $y^2 = x^{-1}$, $x = 1$, $x = 3$; axis $y = -3$

SOLUTION  The region bounded by the graphs of $y^2 = x^{-1}$, $x = 1$ and $x = 3$ is shown below. Let's choose to slice the region vertically. Because a vertical slice is perpendicular to the axis of rotation, we will use the washer method to calculate the volume of the solid of revolution. For each $x \in [1, 3]$, the cross section is a washer with outer radius $R = \frac{1}{\sqrt{x}} - (-3) = 3 + \frac{1}{\sqrt{x}}$ and inner radius $r = -\frac{1}{\sqrt{x}} - (-3) = 3 - \frac{1}{\sqrt{x}}$. The volume of the solid is therefore given by

$$\pi \int_1^3 \left( \left( 3 + \frac{1}{\sqrt{x}} \right)^2 - \left( 3 - \frac{1}{\sqrt{x}} \right)^2 \right) \, dx = 12\pi \int_1^3 x^{-1/2} \, dx = 24\pi \sqrt{x} \bigg|_1^3 = 24\pi(\sqrt{3} - 1).$$

33. $y = -x^2 + 4x - 3$, $y = 0$; axis $y = -1$

SOLUTION  The region bounded by the graph of $y = -x^2 + 4x - 3$ and the $x$-axis is shown below. Let's choose to slice the region vertically. Because a vertical slice is perpendicular to the axis of rotation, we will use the washer method to calculate the volume of the solid of revolution. For each $x \in [1, 3]$, the cross section is a washer with outer radius $R = -x^2 + 4x - 3 - (-1) = -x^2 + 4x - 2$ and inner radius $r = 0 - (-1) = 1$. The volume of the solid is therefore given by

$$\pi \int_1^3 \left( (-x^2 + 4x - 2)^2 - 1 \right) \, dx = \pi \left( \left. \frac{1}{5}x^5 - 2x^4 + \frac{20}{3}x^3 - 8x^2 + 3x \right|_1^3 \right) = \pi \left( \left. \frac{243}{5} - 162 + 180 - 72 + 9 \right) - \left. \left( \frac{1}{5} - 2 + \frac{20}{3} - 8 + 3 \right) \right|_1^3 = \frac{56\pi}{15}.$$ 

34. $x = 4y - y^3$, $y = 0$, $y = 2$; $y$-axis

SOLUTION  The region bounded by the graphs of $x = 4y - y^3$, $y = 0$ and $y = 2$ is shown below. Let's choose to slice this region horizontally. Because a horizontal slice is perpendicular to the axis of rotation, we will use the washer
method to calculate the volume of the solid of revolution. For each \( y \in [0, 2] \), the cross section is a circular disk with radius \( R = 4y - y^3 \). The volume of the solid is therefore given by

\[
\pi \int_0^2 (4y - y^3)^2 \, dy = \pi \left( \frac{16}{3} y^3 - \frac{8}{5} y^5 + \frac{1}{7} y^7 \right)_{0}^{2} = \pi \left( \frac{128}{3} - \frac{256}{5} + \frac{128}{7} \right) = \frac{1024\pi}{105}.
\]

35. \( y^2 = x^{-1}, \ x = 1, \ x = 3; \ \)axis \( x = -3 \)

**SOLUTION** The region bounded by the graphs of \( y^2 = x^{-1}, \ x = 1 \) and \( x = 3 \) is shown below. Let’s choose to slice the region vertically. Because a vertical slice is parallel to the axis of rotation, we will use the shell method to calculate the volume of the solid of revolution. For each \( x \in [1, 3] \), the shell has radius \( x - (-3) = x + 3 \) and height

\[
\frac{1}{\sqrt{x}} - \left( -\frac{1}{\sqrt{x}} \right) = \frac{2}{\sqrt{x}}.
\]

The volume of the solid is therefore given by

\[
2\pi \int_1^3 (x + 3) \frac{2}{\sqrt{x}} \, dx = 2\pi \left( \frac{4}{3} x^{3/2} + 12x^{1/2} \right)_{1}^{3} = 2\pi \left( 16\sqrt{3} - \frac{40}{3} \right).
\]

In Exercises 36–38, the regions refer to the graph of the hyperbola \( y^2 - x^2 = 1 \) in Figure 3. Calculate the volume of revolution about both the \( x \)- and \( y \)-axes.

36. The shaded region between the upper branch of the hyperbola and the \( x \)-axis for \(-c \leq x \leq c \).

**SOLUTION**

- \( x \)-axis: Let’s choose to slice the region vertically. Because a vertical slice is perpendicular to the axis of rotation, we will use the washer method to calculate the volume of the solid of revolution. For each \( x \in [-c, c] \), cross sections are circular disks with radius \( R = \sqrt{1 + x^2} \). The volume of the solid is therefore given by

\[
\pi \int_{-c}^{c} (1 + x^2) \, dx = \pi \left( x + \frac{1}{3} x^3 \right)_{-c}^{c} = \pi \left[ (c + \frac{c^3}{3}) - \left( -c - \frac{c^3}{3} \right) \right] = 2\pi \left( c + \frac{c^3}{3} \right).
\]

- \( y \)-axis: Let’s choose to slice the region vertically. Because a vertical slice is parallel to the axis of rotation, we will use the shell method to calculate the volume of the solid of revolution. For each \( x \in [0, c] \), the shell has radius \( x \) and height \( \sqrt{1 + x^2} \). The volume of the solid is therefore given by

\[
2\pi \int_0^c x \sqrt{1 + x^2} \, dx = \frac{2\pi}{3} \left( 1 + x^2 \right)^{3/2} \bigg|_{0}^{c} = \frac{2\pi}{3} \left( (1 + c^2)^{3/2} - 1 \right).
\]
37. The region between the upper branch of the hyperbola and the line \( y = x \) for \( 0 \leq x \leq c \).

**SOLUTION**

- **x-axis:** Let’s choose to slice the region vertically. Because a vertical slice is perpendicular to the axis of rotation, we will use the washer method to calculate the volume of the solid of revolution. For each \( x \in [0, c] \), cross sections are washers with outer radius \( R = \sqrt{1 + x^2} \) and inner radius \( r = x \). The volume of the solid is therefore given by

\[
\pi \int_0^c \left( (1 + x^2) - x^2 \right) \, dx = \pi \left. x \right|_0^c = c\pi.
\]

- **y-axis:** Let’s choose to slice the region vertically. Because a vertical slice is parallel to the axis of rotation, we will use the shell method to calculate the volume of the solid of revolution. For each \( x \in [0, c] \), the shell has radius \( x \) and height \( \sqrt{1 + x^2} - x \). The volume of the solid is therefore given by

\[
2\pi \int_0^c x \left( \sqrt{1 + x^2} - x \right) \, dx = \frac{2\pi}{3} \left( 1 + x^2 \right)^{3/2} - x^3 \bigg|_0^c = \frac{2\pi}{3} \left( (1 + c^2)^{3/2} - c^3 - 1 \right).
\]

38. The region between the upper branch of the hyperbola and \( y = 2 \).

**SOLUTION** The upper branch of the hyperbola and the horizontal line \( y = 2 \) intersect when \( x = \pm \sqrt{3} \).

- **x-axis:** Using the washer method, cross sections are washers with outer radius \( R = 2 \) and inner radius \( r = \sqrt{1 + x^2} \). The volume of the solid is therefore given by

\[
\pi \int_{-\sqrt{3}}^{\sqrt{3}} \left( 4 - (1 + x^2) \right) \, dx = \pi \left( 3x - \frac{1}{3} x^3 \right) \bigg|_{-\sqrt{3}}^{\sqrt{3}} = \pi \left( (3\sqrt{3} - \sqrt{3}) - (-3\sqrt{3} + \sqrt{3}) \right) = 4\pi\sqrt{3}.
\]

- **y-axis:** Using the shell method, each shell has radius \( x \) and height \( 2 - \sqrt{1 + x^2} \). The volume of the solid is therefore given by

\[
2\pi \int_0^{\sqrt{3}} x \left( 2 - \sqrt{1 + x^2} \right) \, dx = 2\pi \left( x^2 - \frac{1}{3} (1 + x^2)^{3/2} \right) \bigg|_0^{\sqrt{3}} = 2\pi \left( 3 - \frac{8}{3} + \frac{1}{3} \right) = \frac{4\pi}{3}.
\]

39. Let \( a > 0 \). Show that when the region between \( y = a\sqrt{x} - ax^2 \) and the x-axis is rotated about the x-axis, the resulting volume is independent of the constant \( a \).

**SOLUTION** Setting \( a\sqrt{x} - ax^2 = 0 \) yields \( x = 0 \) and \( x = 1/a \). Using the washer method, cross sections are circular disks with radius \( R = a\sqrt{x} - ax^2 \). The volume of the solid is therefore given by

\[
\pi \int_0^{1/a} a^2(x - ax^2) \, dx = \pi \left( \frac{1}{2} a^2 x^2 - \frac{1}{3} a^3 x^3 \right) \bigg|_0^{1/a} = \pi \left( \frac{1}{2} - \frac{1}{3} \right) = \frac{\pi}{6},
\]

which is independent of the constant \( a \).

40. A spring whose equilibrium length is 15 cm exerts a force of 50 N when it is stretched to 20 cm. Find the work required to stretch the spring from 22 to 24 cm.

**SOLUTION** A force of 50 N is exerted when the spring is stretched 5 cm = 0.05 m from its equilibrium length; therefore, the value of the spring constant is \( k = 1000 \) N/m. The work required to stretch the spring from a length of 22 cm to a length of 24 cm is then

\[
\int_{0.07}^{0.09} 1000x \, dx = 500x^2 \bigg|_{0.07}^{0.09} = 500(0.09^2 - 0.07^2) = 1.6 \text{ J}.
\]

In Exercises 41–42, water is pumped into a spherical tank of radius 5 ft from a source located 2 ft below a hole at the bottom (Figure 4). The density of water is 64.2 lb/ft³.
41. Calculate the work required to fill the tank.

**SOLUTION** Place the origin at the base of the sphere with the positive \( y \)-axis pointing upward. The equation for the great circle of the sphere is then \( x^2 + (y - 5)^2 = 25 \). At location \( y \), the horizontal cross section is a circle of radius \( \sqrt{25 - (y - 5)^2} = \sqrt{10y - y^2} \); the volume of the layer is then \( \pi (10y - y^2) \Delta y \) ft\(^3\), and the force needed to lift the layer is \( 64.2\pi (10y - y^2) \Delta y \) lb. The layer of water must be lifted \( y + 2 \) feet, so the work required to fill the tank is given by

\[
64.2\pi \int_{0}^{10} (y + 2)(10y - y^2) \, dy = 64.2\pi \int_{0}^{10} (8y^2 + 20y - y^3) \, dy
\]

\[
= 64.2\pi \left( \frac{8}{3}y^3 + 10y^2 - \frac{1}{4}y^4 \right)_{0}^{10}
\]

\[
= 74900\pi \approx 235,305 \text{ ft-lb}.
\]

42. Calculate the work \( F(h) \) required to fill the tank to height \( h \) ft from the bottom of the sphere.

**SOLUTION** Place the origin at the base of the sphere with the positive \( y \)-axis pointing upward. The equation for the great circle of the sphere is then \( x^2 + (y - 5)^2 = 25 \). At location \( y \), the horizontal cross section is a circle of radius \( \sqrt{25 - (y - 5)^2} = \sqrt{10y - y^2} \); the volume of the layer is then \( \pi (10y - y^2) \Delta y \) ft\(^3\), and the force needed to lift the layer is \( 64.2\pi (10y - y^2) \Delta y \) lb. The layer of water must be lifted \( y + 2 \) feet, so the work required to fill the tank is given by

\[
64.2\pi \int_{0}^{h} (y + 2)(10y - y^2) \, dy = 64.2\pi \int_{0}^{h} (8y^2 + 20y - y^3) \, dy
\]

\[
= 64.2\pi \left( \frac{8}{3}y^3 + 10y^2 - \frac{1}{4}y^4 \right)_{0}^{h}
\]

\[
= 64.2\pi \left( \frac{8}{3}h^3 + 10h^2 - \frac{1}{4}h^4 \right) \text{ ft-lb}.
\]

43. A container weighing 50 lb is filled with 20 ft\(^3\) of water. The container is raised vertically at a constant speed of 2 ft/s for 1 min, during which time it leaks water at a rate of \( \frac{1}{3} \) ft\(^3\)/s. Calculate the total work performed in raising the container. The density of water is 64.2 lb/ft\(^3\).

**SOLUTION** Let \( t \) denote the elapsed time of the ascent of the container, and let \( y \) denote the height of the container. Given that the speed of ascent is 2 ft/s, \( y = 2t \); moreover, the volume of water in the container is

\[
20 - \frac{1}{3}t = 20 - \frac{1}{6}y \text{ ft}^3.
\]

The force needed to lift the container and its contents is then

\[
50 + 64.2 \left( 20 - \frac{1}{6}y \right) = 1334 - 10.7y \text{ lb},
\]

and the work required to lift the container and its contents is

\[
\int_{0}^{120} (1334 - 10.7y) \, dy = (1334y - 5.35y^2)|_{0}^{120} = 83040 \text{ ft-lb}.
\]

44. Let \( W \) be the work (against the sun’s gravitational force) required to transport an 80-kg person from Earth to Mars when the two planets are aligned with the sun at their minimal distance of 55.7 \( \times \) 10\(^6\) km. Use Newton’s Universal Law of Gravity (see Exercises 32–34 in Section 6.5) to express \( W \) as an integral and evaluate it. The sun has mass \( M_s = 1.99 \times 10^{30} \) kg, and the distance from the sun to the earth is 149.6 \( \times \) 10\(^6\) km.

**SOLUTION** According to Newton’s Universal Law of Gravity, the gravitational force between the person and the sun is

\[
\frac{G M_s m}{r^2},
\]

where \( G = 6.67 \times 10^{-11} \text{ m}^3\text{kg}^{-1}\text{s}^{-2} \) is a constant, \( M_s = 1.99 \times 10^{30} \) kg is the mass of the sun, \( m = 80 \) kg is the mass of the person, and \( r \) is the distance between the sun and the person. The work against the sun’s gravitational force required to transport the person from Earth to Mars when the two planets are aligned with the sun is therefore given by

\[
W = \int_{r_{se}}^{r_{se}+r_{em}} \frac{G M_s m}{r^2} \, dr = G M_s m \left( \frac{1}{r_{se}} - \frac{1}{r_{se}+r_{em}} \right),
\]
where \( r_{se} = 149.6 \times 10^6 \text{ km} \) is the distance from the sun to Earth and \( r_{em} = 55.7 \times 10^6 \text{ km} \) is the distance from Earth to Mars. Converting the distances to meters and substituting the known values into the formula for \( W \) yields

\[
W = (6.67 \times 10^{-11})(1.99 \times 10^{30})(80) \left( \frac{1}{149.6 \times 10^9} - \frac{1}{205.3 \times 10^9} \right) \approx 1.93 \times 10^{10} \text{ J}.
\]