SIMPLE HARMONIC MOTION

I. INTRODUCTION

A. ANY SYSTEM IN STABLE EQUILIBRIUM WILL OSCILLATE WITH HARMONIC MOTION ABOUT THE EQUILIBRIUM POINT IF WE DISTURB IT. RECALL, AN OBJECT IS IN EQUILIBRIUM IF NO FORCE ACTS ON IT. STABLE EQUILIBRIUM MEANS THAT IF WE DISTURB AN OBJECT, A RESTORING FORCE WILL TRY TO PUSH OR PULL THE OBJECT BACK TO THE EQUILIBRIUM POSITION.

B. BASIC FORMULAS

$x = A\cos(\omega t)$ [m] DISPLACEMENT, LOCATION

$v = -A\omega\sin(\omega t)$ [m/s] VELOCITY

$a = -A\omega^2\cos(\omega t)$ [m/s²] ACCELERATION

$F = ma = -mA\omega^2\cos(\omega t)$ [N] FORCE

ω = ANGULAR FREQUENCY [RADIANS/sec]

$f = \omega/2\pi$ = FREQUENCY [CYCLES/SEC] = [HZ]

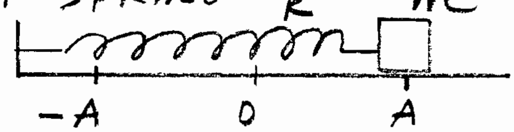
$T = 1/f = 2\pi/\omega$ = PERIOD = [sec/CYCLE]

A = AMPLITUDE [m]

THE FREQUENCY OF OSCILLATION ONLY DEPENDS UPON THE PHYSICAL CHARACTERISTICS OF THE SYSTEM. THIS FREQUENCY, CALLED THE NATURAL FREQUENCY OF AN OBJECT, IS THE FREQUENCY AT WHICH AN OBJECT WOULD PREFER TO VIBRATE. IF WE DISTURB AN OBJECT IN STABLE EQUILIBRIUM AND RELEASE IT, THE OBJECT WILL VIBRATE AT ITS NATURAL FREQUENCY.

II. MASS ON A SPRING

A. BASICS:



$\omega = \sqrt{K/m}$ K = SPRING CONSTANT   m = MASS

$f = \frac{1}{2\pi}\sqrt{K/m}$ $T = 2\pi\sqrt{m/K}$

INTUITIONS: LARGE MASS / STRONG SPRING } YIELDS LONG PERIOD.

EXAMPLE: GIVEN = m = 12,665 KG K = 20 N/m.

FIND THE FREQUENCY AND PERIOD OF THE MOTION.

$f = \frac{1}{2\pi}\sqrt{\frac{20}{12,665}} = .2\ Hz$ $T = \frac{1}{f} = \frac{5\ sec}{CYCLE}$

FIND THE TIME REQUIRED TO MAKE 40 CYCLES.

$t = 40\ cycles\left(\frac{5\ sec}{cycle}\right) = 200\ sec.$

FIND THE CYCLES MAKE IN 320 SEC.

$Cycles = 320\ sec\left(\frac{.2\ cycles}{sec}\right) = 64\ cycles$

B. EQUATIONS OF MOTION

$ma = F_s$

$ma = -KX$

THE NEGATIVE SIGN INDICATES A RESTORING FORCE.

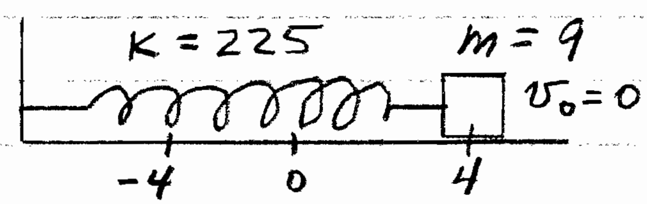
THE SOLUTION OF THIS EQUATION IS:

$x = A\cos\omega t$

$v = -A\omega\sin\omega t$

$a = -A\omega^2\cos\omega t$

EXAMPLE: GIVEN:



WE RELEASE THE MASS FROM REST.

FIND ω. $\omega = \sqrt{K/m} = 5\ sec^{-1}$

$\therefore x = 4\cos 5t$

$v = -20\sin 5t$

$a = -100\cos 5t$

CALCULATORS IN RADIANS.

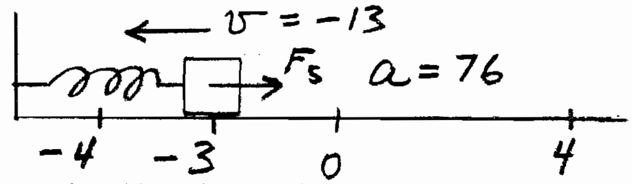
CALCULATE x, v AND a AT $t = 3$ SECONDS.

$x = 4\cos[(5)(3)] = -3.04$ m

$v = -20 \sin 15 = -13$ m/s

$a = -100 \cos 15 = 76$ m/s^2

CARTOON AT $t = 3$.



SUPPLEMENTARY INFORMATION.

$F = -kx \quad \therefore \quad F = -k(A\cos\omega t)$

$F = ma \quad \therefore \quad F = m(-A\omega^2 \cos \omega t)$

$KE = \frac{1}{2} m v^2 = \frac{1}{2} m[-A\omega \sin \omega t]^2$

$PE = \frac{1}{2} k x^2 = \frac{1}{2} k [A \cos \omega t]^2$

CONTINUING THE EXAMPLE ABOVE: (SEE PREVIOUS PAGE)

FIND THE LOCATION AT WHICH THE SPEED IS 15.

$PE_o + KE_o = PE + KE \qquad \frac{1}{2}kA^2 + 0 = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$

$\frac{1}{2}(225)(4)^2 + 0 = \frac{1}{2}(225)x^2 + \frac{1}{2}(9)(15)^2$

$1800 = \frac{1}{2}(225)x^2 + 1012.5 \qquad x = 2.65$ m/s

NOTE: BY CONSERVATION OF ENERGY, THE TOTAL ENERGY IS CONSTANT. AT ANY TIME AND AT ANY POINT, $PE + KE = E_{TOTAL}$.

SINCE THE MASS STARTED AT REST,

$E_{TOTAL} = PE_o + KE_o = \frac{1}{2}kA^2 + 0 = \frac{1}{2}kA^2$.

AT OTHER TIMES,

$PE + KE = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 =$

$\frac{1}{2}k(A\cos\omega t)^2 + \frac{1}{2}m(-A\omega\sin\omega t)^2$

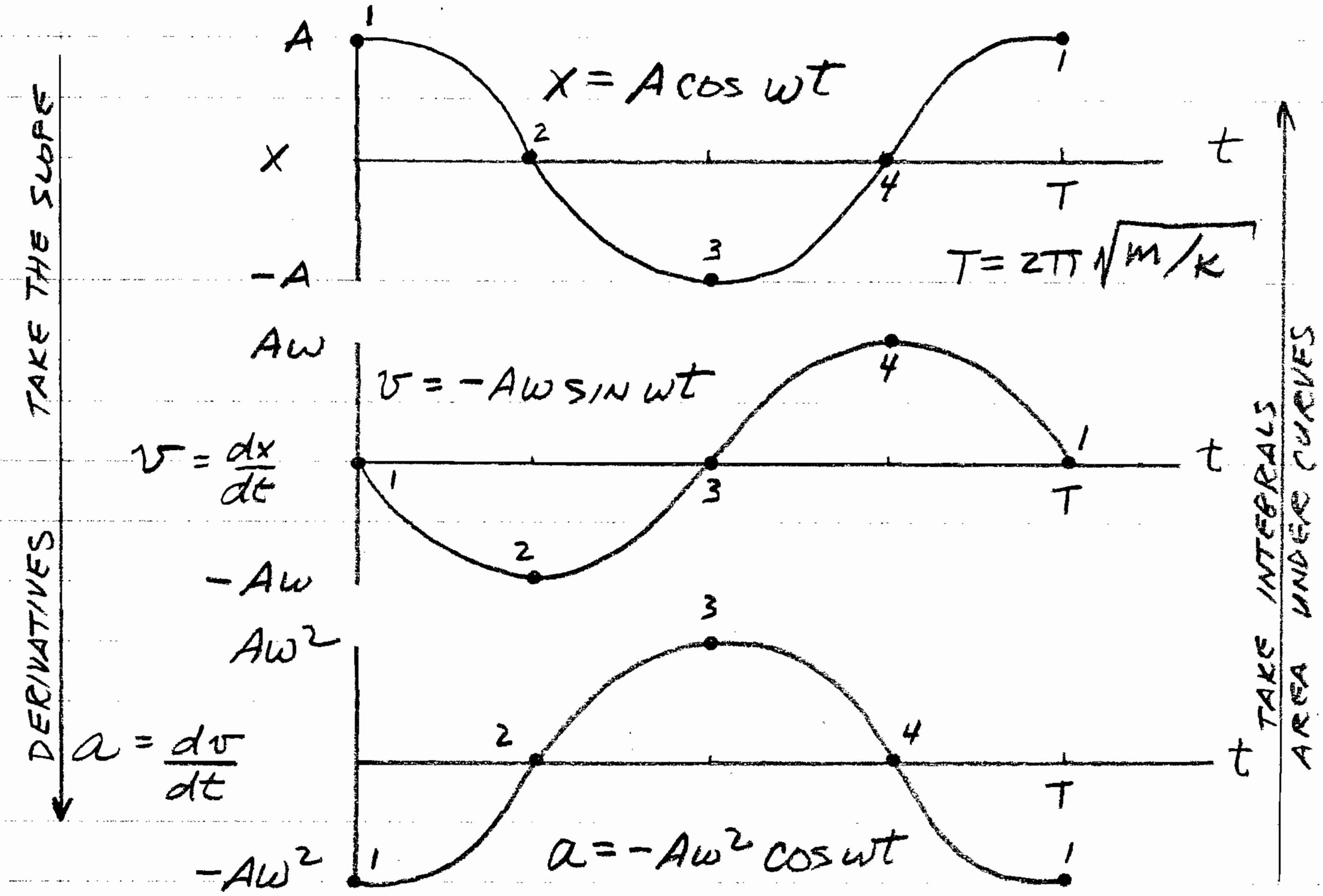
$\frac{1}{2}kA^2\cos^2\omega t + \frac{1}{2}mA^2\omega^2\sin^2\omega t$

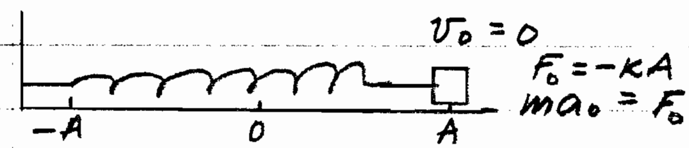
$\frac{1}{2}kA^2\cos^2\omega t + \frac{1}{2}mA^2(k/m)\sin^2\omega t$

$\frac{1}{2}kA^2(\cos^2\omega t + \sin^2\omega t)$

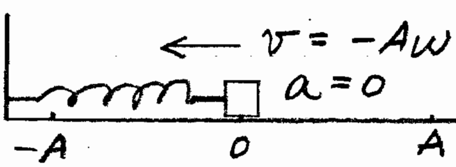
$\frac{1}{2}kA^2$ ✓

C. GRAPHS FOR MASS ON A SPRING

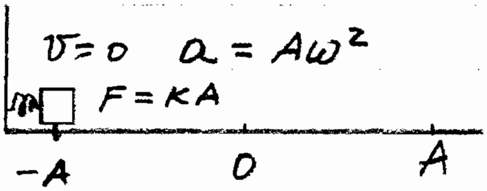


1 

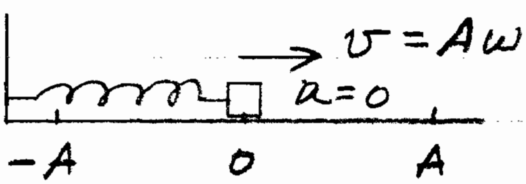
$t = 0$ $t = T$

2 

$t = \frac{1}{4}T$

3 

$t = \frac{1}{2}T$

4 

$t = \frac{3}{4}T$

RECALL: $F = -kx$ $\qquad PE = \frac{1}{2}kx^2$

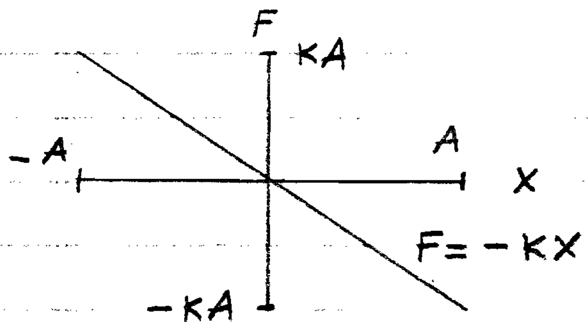
$PE_0 + KE_0 = PE + KE$

$\frac{1}{2}kA^2 + 0 = \frac{1}{2}kx^2 + KE$

$\therefore KE = \frac{1}{2}kA^2 - \frac{1}{2}kx^2$

IN WORDS, THE TOTAL ENERGY OF THE SYSTEM IS $\frac{1}{2}kA^2$, THE ORIGINAL ENERGY WE STORED IN THE SPRING. AFTER WE RELEASE THE MASS, AT LOCATION x, THE PE REMAINING IN THE SPRING IS $\frac{1}{2}kx^2$. THE PE LOST BY THE SPRING HAS BEEN CONVERTED INTO THE KE OF THE MASS.

GRAPHS:

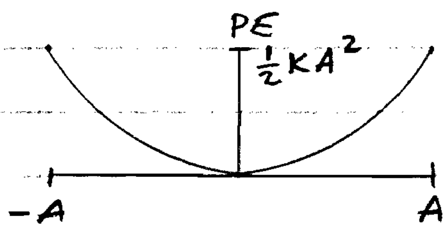


RECALL:

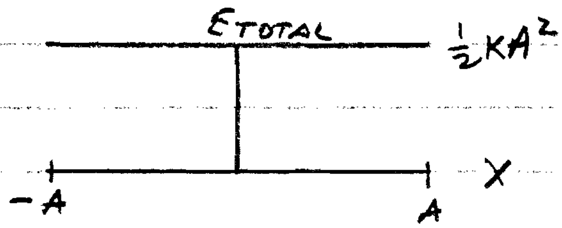
$PE = -\int (F)\,dx$

$PE = -\int (-kx)\,dx$

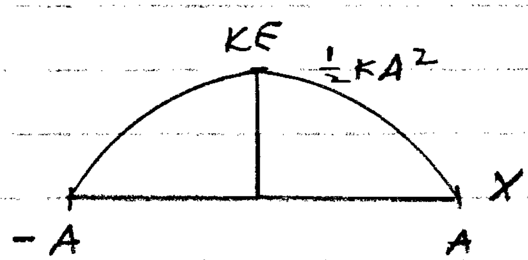
$PE = \frac{1}{2}kx^2$



$PE = \frac{1}{2}kx^2$



$E_{TOTAL} = \frac{1}{2}kA^2$ ALWAYS $\qquad E_{TOTAL} = PE + KE$

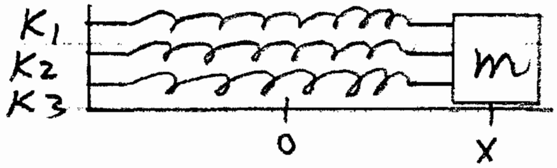


$KE = \frac{1}{2}kA^2 - \frac{1}{2}kx^2$

D. COMBINATIONS OF SPRINGS

1. PARALLEL

$K_{TOTAL} = K_1 + K_2 + K_3$

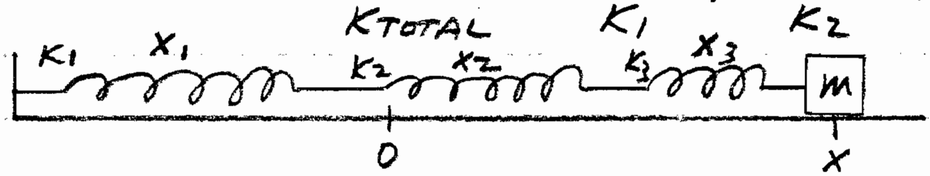


ALL SPRINGS ARE STRETCHED THE SAME DISTANCE X. THE TOTAL FORCE ON THE MASS IS HUGE: $F_1 + F_2 + F_3$. THE TOTAL PE STORED IN THE SPRINGS IS HUGE: $PE_{TOTAL} = PE_1 + PE_2 + PE_3 = \frac{1}{2} K_1 X^2 + \frac{1}{2} K_2 X^2 + \frac{1}{2} K_3 X^2$. THESE ARE THE REASONS THAT THE NET SPRING CONSTANT IS HUGE. FOR THE RESULTING HARMONIC MOTION:

$$\omega = \sqrt{\frac{K_{TOTAL}}{m}} \qquad T = 2\pi \sqrt{\frac{m}{K_{TOTAL}}}$$

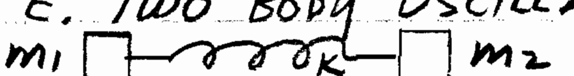
2. SERIES

$$\frac{1}{K_{TOTAL}} = \frac{1}{K_1} + \frac{1}{K_2} + \frac{1}{K_3}$$



EACH SPRING IS ONLY STRETCHED A LITTLE BIT: X_1, X_2 AND X_3, RESPECTIVELY, TO PRODUCE THE TOTAL DISPLACEMENT, X, OF THE MASS. THE FORCE ON THE MASS IS CONSTANT THROUGHOUT THE ENTIRE SPRING SYSTEM. SINCE EVEN THE STRONGEST SPRING IS NOT STRETCHED VERY MUCH, THE FORCE ON THE MASS IS SMALL. FINALLY, SINCE EACH SPRING IS ONLY STRETCHED A LITTLE BIT, THEY ONLY STORE A SMALL AMOUNT OF P.E. ∴ K_{TOTAL} IS SMALL.

E. TWO BODY OSCILLATIONS



$$\frac{1}{m} = \frac{1}{m_1} + \frac{1}{m_2}$$

m = REDUCED MASS

THE FREQUENCY OF THE SYSTEM IS LARGE SINCE THE INERTIA OF THE SYSTEM IS SMALL. A SPRING ATTACHED TO A WALL HAS ONE END WITH INFINITE INERTIA.

$$\omega = \sqrt{K/m}$$

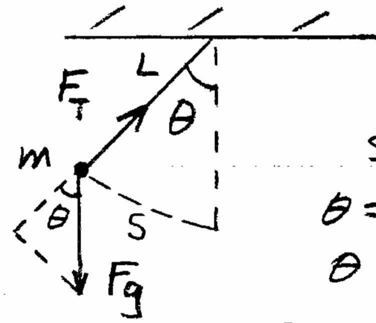
$$T = 2\pi \sqrt{m/K}$$

III. PENDULUM

A MASS HANGING ON A STRING IN A GRAVITATIONAL FIELD IS IN STABLE EQUILIBRIUM. IF WE DISTURB THE MASS, IT WILL OSCILLATE IN HARMONIC MOTION ABOUT THE EQUILIBRIUM POINT.

A. CARTOON:

$S = L\theta$



L = LENGTH [m]
S = ARC LENGTH [m]
θ = DISPLACEMENT
θ = [RADIANS]

THE RESTORING FORCE IS $F_g \sin\theta$ OR $mg\sin\theta$. THE NET INWARD FORCE, $F_T - F_g\cos\theta$, CAUSES THE MASS TO ACCELERATE BY CHANGING THE DIRECTION OF MOTION. $F_g \sin\theta$ CAUSE ACCELERATION BY CHANGING THE SPEED OF THE MASS.

B. FORMULAS

$\omega = \sqrt{g/L}$ [sec^{-1}] $f = \frac{1}{2\pi}\sqrt{g/L}$ [HERTZ]

$T = 2\pi\sqrt{L/g}$ [sec/CYCLE] $g = 10\ m/s^2$

INTUITIONS:

1. LONG PENDULUMS HAVE LONG PERIODS.
2. PLANETS WITH SMALL GRAVITY HAVE PENDULUMS WITH LONG PERIODS.
3. MASS HAS NO INFLUENCE ON THE PERIOD.

IF THE MASS IS LARGE, F_g IS LARGE. BUT LARGE MASS HAS A LOT OF INERTIA. ∴ IT NEEDS A LARGE F_g TO ACCELERATE. THESE TWO OPPOSING EFFECTS EXACTLY CANCEL OUT.

4. THE ANGLE AT WHICH WE START THE PENDULUM HAS NO INFLUENCE ON THE PERIOD. IF WE START AT A LARGE ANGLE, THE MASS HAS

A LOT OF P.E. THUS, THE MASS WILL HAVE A LOT OF KE AND IT WILL MOVE FAST. BUT IT NEEDS TO MOVE FAST SINCE IT HAS A FURTHER DISTANCE TO TRAVEL. THESE TWO OPPOSING EFFECTS EXACTLY CANCEL OUT.

EXAMPLE: A PENDULUM 6 m LONG HAS A PERIOD OF 5 SECONDS. FIND THE PERIOD OF A PENDULUM 12 m LONG. $T = 2\pi\sqrt{L/g}$

$$\frac{T_2}{T_1} = \frac{2\pi\sqrt{L_2/g}}{2\pi\sqrt{L_1/g}} = \sqrt{\frac{L_2}{L_1}}$$

$$T_2 = T_1\sqrt{L_2/L_1} = 5\sqrt{12/6} = 7 \text{ SECONDS.}$$

EXAMPLE: ON EARTH, A PENDULUM HAS PERIOD OF 6 SECONDS. ON A NEW PLANET, WOLFSON, THIS SAME PENDULUM HAS PERIOD 1.5 SECONDS. FIND THE GRAVITY ON WOLFSON.

$$\frac{T_2}{T_1} = \frac{2\pi\sqrt{L/g_2}}{2\pi\sqrt{L/g_1}} = \sqrt{\frac{g_1}{g_2}} \quad \therefore \frac{T_2^2}{T_1^2} = \frac{g_1}{g_2}$$

$$g_2 = g_1(T_1/T_2)^2 = 10(6/1.5)^2 = 160 \text{ m/s}^2$$

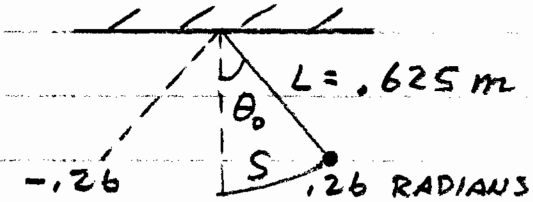
C. EQUATIONS OF MOTION

$$\left.\begin{aligned} \theta &= \theta_0 \cos\omega t \quad [\text{RADIANS}] \\ S &= S_0 \cos\omega t = L\theta_0 \cos\omega t \quad [\text{m}] \end{aligned}\right\} \text{LOCATION}$$

$$\left.\begin{aligned} \Omega &= -\theta_0\omega \sin\omega t \quad [\text{RADIANS/SEC}] \\ v &= -L\theta_0\omega \sin\omega t \quad [\text{m/s}] \end{aligned}\right\} \text{SPEED}$$

$$\left.\begin{aligned} \alpha &= -\theta_0\omega^2 \cos\omega t \quad [\text{sec}^{-2}] \\ a_T &= -L\theta_0\omega^2 \cos\omega t \quad [\text{m/s}^2] \end{aligned}\right\} \text{CHANGE IN SPEED}$$

EXAMPLE: A PENDULUM HAS LENGTH .625 m. PAUL PULLS IT ASIDE TO ANGULAR LOCATION .26 RADIANS AND RELEASES IT FROM REST.



a) FIND THE ANGULAR FREQUENCY, FREQUENCY AND PERIOD OF OSCILLATION.

$\omega = \sqrt{g/L} = \sqrt{10/.625} = 4 \text{ sec}^{-1}$

$f = \omega/2\pi = \frac{1}{2\pi}\sqrt{g/L} = .6366 \text{ Hz}$

$T = 1/f = 2\pi\sqrt{L/g} = 1.57$ SECONDS/cycle

b) WRITE EQUATIONS FOR θ, Ω AND α.

$\theta = .26 \cos 4t$ LOCATION IN RADIANS

$\Omega = -1.04 \sin 4t$ ANGULAR SPEED IN $\frac{\text{RADIANS}}{\text{SEC}}$

$\alpha = -4.16 \cos 4t$ ANGULAR ACCELERATION IN SEC^{-2}.

c) WRITE EQUATIONS FOR S, v AND a_T.

$S = .1625 \cos 4t$ LOCATION IN METERS

$v = -.65 \sin 4t$ SPEED IN m/s

$a_T = -2.6 \cos 4t$ TANGENTIAL ACCELERATION IN m/s^2

d) CALCULATE THE LOCATION, SPEED AND TANGENTIAL ACCELERATION OF THE MASS WHEN $t = .6$ SEC.

$S = .1625 \cos(4(.6)) = -.12 \text{ m}$

$v = -.65 \sin(2.4) = -.44 \text{ m/s}$

$a_T = -2.6 \cos(2.4) = 1.917 \text{ m/s}^2$

AT $t = .6$.